

The Thomsen model of inserts in sandwich composites: An evaluation

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September 28, 2010

Abstract

A one-dimensional finite element model of a sandwich panel with insert is derived using the approach used in the Thomsen model. The one-dimensional model produces results that are close to those of a two-dimensional axisymmetric model. Both models assume that the core is homogeneous. Our results indicate that the one-dimensional model may be well suited for small deformations of sandwich specimens with foam cores.

1 Introduction

The numerical simulation of complicated sandwich structures containing inserts can be computationally expensive, particularly when a statistical analysis of the effect of variable input parameters is the goal. Simplified theories of sandwich structures provide a means of assessing the adequacy of the particular statistical technique that is of interest.

Theories of sandwich structures can be broadly classified into the following types:

- First-order theories (see for example, [1]).
- Higher-order linear theories that do not account for thickness change (see for example [2] and references therein).
- Geometrically-exact single-layer nonlinear theories that do not account for thickness change (see for example, [3]).
- Higher-order linear single-layer theories that account for thickness change (see for example, [4, 5, 6]).
- Higher-order linear multi-layer theories that account for thickness change (see for example, [7, 8, 9, 10, 11]).
- Higher-order nonlinear single-layer theories that account for thickness change (see for example, [12, 13, 14, 15, 16]).

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Most theories start with ad-hoc assumptions about the displacement or stress field. Geometrically-exact theories avoid such assumptions but are hampered by the requirement that special constitutive models have to be designed for consistency. The linear theory proposed by Thomsen and co-workers [7, 8, 9] provides a formulation that is simple enough to be evaluated rapidly. Therefore, we have chosen that formulation and applied it to an axisymmetric sandwich panel in this work. The work of Thomsen involves the solution of a system of first order ordinary differential equations using a multi-segment numerical method, We have instead chosen to use the considerably simpler finite element method to discretize and solve the system of equations.

2 The Thomsen Model

Since we are considering a simplified axisymmetric form of the sandwich panel problem, we start with the governing equations expressed in cylindrical coordinates. The geometry of the sandwich structure under consideration is shown in Figure 1.

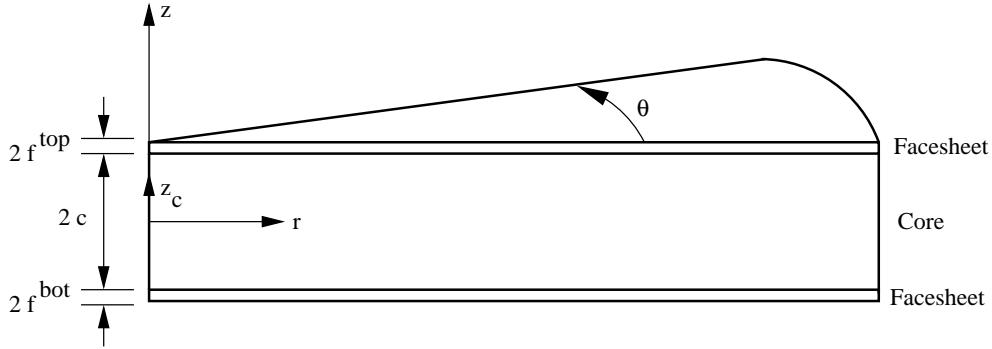


Figure 1 – The geometry of the sandwich panel.

2.0.1 Strain-displacement

The strain-displacement relations are given by

$$\epsilon = \frac{1}{2} [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \quad (1)$$

In cylindrical coordinates we have

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial u_r}{\partial r} ; \quad \epsilon_{\theta\theta} = \frac{1}{r} \left(\frac{\partial u_\theta}{\partial \theta} + u_r \right) ; \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z} \\ \epsilon_{r\theta} &= \frac{1}{2} \left[\frac{\partial u_\theta}{\partial r} + \frac{1}{r} \left(\frac{\partial u_r}{\partial \theta} - u_\theta \right) \right] ; \quad \epsilon_{\theta z} = \frac{1}{2} \left[\frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right] ; \quad \epsilon_{rz} = \frac{1}{2} \left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right] \end{aligned} \quad (2)$$

Axisymmetry implies that the displacement $u_\theta = u_\theta(r)$ and all derivatives with respect to θ are zero. If in addition, the displacements are small such that $u_\theta = C r$ (this assumption is not strictly necessary), the strain-displacement relations reduce to

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial u_r}{\partial r} ; \quad \epsilon_{\theta\theta} = \frac{u_r}{r} ; \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z} \\ \epsilon_{\theta z} &= 0 ; \quad \epsilon_{rz} = \frac{1}{2} \left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right] ; \quad \epsilon_{r\theta} = 0 \end{aligned} \quad (3)$$

2.0.2 Stress-strain

The stress-strain relations for an orthotropic material are

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\epsilon} \quad (4)$$

In cylindrical coordinates

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{\theta z} \\ \sigma_{rz} \\ \sigma_{r\theta} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \varepsilon_{\theta z} \\ \varepsilon_{rz} \\ \varepsilon_{r\theta} \end{bmatrix} = \quad (5)$$

From axisymmetry, we therefore have

$$\begin{aligned} \sigma_{rr} &= C_{11} \varepsilon_{rr} + C_{12} \varepsilon_{\theta\theta} + C_{13} \varepsilon_{zz} \\ \sigma_{\theta\theta} &= C_{12} \varepsilon_{rr} + C_{22} \varepsilon_{\theta\theta} + C_{23} \varepsilon_{zz} \\ \sigma_{zz} &= C_{13} \varepsilon_{rr} + C_{23} \varepsilon_{\theta\theta} + C_{33} \varepsilon_{zz} \\ \sigma_{\theta z} &= 0 ; \quad \sigma_{rz} = C_{55} \varepsilon_{rz} ; \quad \sigma_{r\theta} = 0 \end{aligned} \quad (6)$$

2.0.3 Equilibrium

We assume that there are no inertial or body forces in the sandwich panel. Then the three-dimensional equilibrium equations take the form

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0} \quad (7)$$

The equilibrium equations in cylindrical coordinates are

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \left[\frac{\partial \sigma_{r\theta}}{\partial \theta} + (\sigma_{rr} - \sigma_{\theta\theta}) \right] + \frac{\partial \sigma_{rz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \left[\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + 2\sigma_{r\theta} \right] + \frac{\partial \sigma_{\theta z}}{\partial z} &= 0 \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \left[\frac{\partial \sigma_{\theta z}}{\partial \theta} + \sigma_{rz} \right] + \frac{\partial \sigma_{zz}}{\partial z} &= 0 \end{aligned} \quad (8)$$

Because of axisymmetry, all derivatives with respect to θ are zero and also $\sigma_{\theta z}$ and $\sigma_{r\theta}$ are zero, the reduced equilibrium equations are

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} [\sigma_{rr} - \sigma_{\theta\theta}] + \frac{\partial \sigma_{rz}}{\partial z} &= 0 \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \sigma_{rz} + \frac{\partial \sigma_{zz}}{\partial z} &= 0 \end{aligned} \quad (9)$$

2.1 Facesheet equations

The facesheets are modeled using the Kirchhoff-Love hypothesis, i.e., that transverse normals remain straight and normal and that the normals are inextensible. In that case, the displacement field

in the plate takes the form:

$$u_r(r, \theta, z) = u_{0r}(r, \theta) - z \frac{\partial w_0}{\partial r} ; \quad u_\theta(r, \theta, z) = u_{0\theta}(r, \theta) - z \frac{\partial w_0}{\partial \theta} ; \quad u_z(r, \theta, z) = w_0(r, \theta) \quad (10)$$

where u_{0r} is the displacement of the midsurface in the r -direction, $u_{0\theta}$ is the displacement of the midsurface in the θ -direction, and w_0 is the z -direction displacement of the midsurface.

We define the stress resultants and stress couples as

$$N_{rr} := \int_{-f}^f \sigma_{rr} dz ; \quad N_{\theta\theta} := \int_{-f}^f \sigma_{\theta\theta} dz ; \quad M_{rr} := \int_{-f}^f z \sigma_{rr} dz ; \quad M_{\theta\theta} := \int_{-f}^f z \sigma_{\theta\theta} dz \quad (11)$$

where the thickness of the plate is $2f$.

2.1.1 Strain-displacement relations

From axisymmetry, the strain-displacement relations are (for small rotations, i.e., NOT the von Karman strains)

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} ; \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} ; \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} ; \quad \varepsilon_{\theta z} = 0 ; \quad \varepsilon_{rz} = \frac{1}{2} \left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right] ; \quad \varepsilon_{r\theta} = 0 \quad (12)$$

Plugging in the displacement functions in the strain-displacement relations gives

$$\begin{aligned} \varepsilon_{rr} &= \frac{du_{0r}}{dr} - z \frac{d^2 w_0}{dr^2} ; \quad \varepsilon_{\theta\theta} = \frac{u_{0r}}{r} - \frac{z}{r} \frac{dw_0}{dr} ; \quad \varepsilon_{zz} = 0 \\ \varepsilon_{\theta z} &= 0 ; \quad \varepsilon_{rz} = \frac{1}{2} \left[-\frac{dw_0}{dr} + \frac{dw_0}{dr} \right] = 0 ; \quad \varepsilon_{r\theta} = 0 \end{aligned} \quad (13)$$

To simplify the notation, we define

$$\begin{aligned} \varepsilon_{rr}^0(r) &:= \frac{du_{0r}}{dr} ; \quad \varepsilon_{rr}^1(r) := -\frac{d^2 w_0}{dr^2} \\ \varepsilon_{\theta\theta}^0(r) &:= \frac{u_{0r}}{r} ; \quad \varepsilon_{\theta\theta}^1(r) := -\frac{1}{r} \frac{dw_0}{dr} \end{aligned} \quad (14)$$

to get

$$\varepsilon_{rr}(r, z) = \varepsilon_{rr}^0(r) + z \varepsilon_{rr}^1(r) ; \quad \varepsilon_{\theta\theta}(r, z) = \varepsilon_{\theta\theta}^0(r) + z \varepsilon_{\theta\theta}^1(r) \quad (15)$$

2.1.2 Stress-strain relations

Assuming that the facesheets are transversely isotropic and taking into account the strain-displacement relations (13), the axisymmetric stress-strain relations are

$$\begin{aligned} \sigma_{rr} &= C_{11} \varepsilon_{rr} + C_{12} \varepsilon_{\theta\theta} ; \quad \sigma_{\theta\theta} = C_{12} \varepsilon_{rr} + C_{11} \varepsilon_{\theta\theta} ; \quad \sigma_{zz} = C_{13} \varepsilon_{rr} + C_{13} \varepsilon_{\theta\theta} \\ \sigma_{\theta z} &= 0 ; \quad \sigma_{rz} = 0 ; \quad \sigma_{r\theta} = 0 \end{aligned} \quad (16)$$

Using the definitions in (14) the stress-strain relations reduce to

$$\begin{aligned} \sigma_{rr} &= C_{11} \varepsilon_{rr}^0 + z C_{11} \varepsilon_{rr}^1 + C_{12} \varepsilon_{\theta\theta}^0 + z C_{12} \varepsilon_{\theta\theta}^1 \\ \sigma_{\theta\theta} &= C_{12} \varepsilon_{rr}^0 + z C_{12} \varepsilon_{rr}^1 + C_{11} \varepsilon_{\theta\theta}^0 + z C_{11} \varepsilon_{\theta\theta}^1 \\ \sigma_{zz} &= C_{13} \varepsilon_{rr}^0 + z C_{13} \varepsilon_{rr}^1 + C_{13} \varepsilon_{\theta\theta}^0 + z C_{13} \varepsilon_{\theta\theta}^1 \end{aligned} \quad (17)$$

If we make the plane stress assumption, $\sigma_{zz} = 0$, then we have

$$\varepsilon_{rr} = -\varepsilon_{\theta\theta} . \quad (18)$$

Then the relations between the stress resultants and stress couples and the strains are

$$\begin{aligned} N_{rr} &= C_{11} \int_{-f}^f \varepsilon_{rr} dz + C_{12} \int_{-f}^f \varepsilon_{\theta\theta} dz \\ N_{\theta\theta} &= C_{12} \int_{-f}^f \varepsilon_{rr} dz + C_{11} \int_{-f}^f \varepsilon_{\theta\theta} dz \\ M_{rr} &= C_{11} \int_{-f}^f z \varepsilon_{rr} dz + C_{12} \int_{-f}^f z \varepsilon_{\theta\theta} dz \\ M_{\theta\theta} &= C_{12} \int_{-f}^f z \varepsilon_{rr} dz + C_{11} \int_{-f}^f z \varepsilon_{\theta\theta} dz \end{aligned} \quad (19)$$

From the expressions for strain in equations (15)

$$\begin{aligned} \int_{-f}^f \varepsilon_{rr}(r, z) dz &= 2f \varepsilon_{rr}^0(r) ; \quad \int_{-f}^f z \varepsilon_{rr}(r, z) dz = \frac{2f^3}{3} \varepsilon_{rr}^1(r) \\ \int_{-f}^f \varepsilon_{\theta\theta}(r, z) dz &= 2f \varepsilon_{\theta\theta}^0(r) ; \quad \int_{-f}^f z \varepsilon_{\theta\theta}(r, z) dz = \frac{2f^3}{3} \varepsilon_{\theta\theta}^1(r) \end{aligned} \quad (20)$$

Therefore, the relations between the stress resultants and stress couples and the strain can be expressed in matrix form as

$$\begin{bmatrix} N_{rr} \\ N_{\theta\theta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{11} \end{bmatrix} \begin{bmatrix} \varepsilon_{rr}^0 \\ \varepsilon_{\theta\theta}^0 \end{bmatrix} \quad (21)$$

and

$$\begin{bmatrix} M_{rr} \\ M_{\theta\theta} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{11} \end{bmatrix} \begin{bmatrix} \varepsilon_{rr}^1 \\ \varepsilon_{\theta\theta}^1 \end{bmatrix} \quad (22)$$

where $A_{ij} = 2f C_{ij}$ are the extensional stiffnesses of the plate and $D_{ij} = 2f^3/3 C_{ij}$ are the bending stiffnesses of the plate.

2.1.3 Equilibrium equations

The plate equilibrium equations may be derived directly from the three-dimensional equilibrium equations. However, it is more informative to derive them from the principle of virtual work

$$\delta U = \delta V_{\text{ext}} \quad (23)$$

where δU is a variation of the internal energy and δV_{ext} is a variation of the work done by external forces.

The variation in the internal energy is given by

$$\delta U = \int_{\Omega_0} \int_{-f}^f [\sigma_{rr} \delta \varepsilon_{rr} + \sigma_{\theta\theta} \delta \varepsilon_{\theta\theta}] dz d\Omega_0 \quad (24)$$

where Ω_0 represents the reference surface of the plate. In terms of the definitions in (14),

$$\delta U = \int_{\Omega_0} \int_{-f}^f [\sigma_{rr} \delta \varepsilon_{rr}^0 + z \sigma_{rr} \delta \varepsilon_{rr}^1 + \sigma_{\theta\theta} \delta \varepsilon_{\theta\theta}^0 + z \sigma_{\theta\theta} \delta \varepsilon_{\theta\theta}^1] dz d\Omega_0 \quad (25)$$

The definitions in (11) give

$$\delta U = \int_{\Omega_0} [N_{rr} \delta \varepsilon_{rr}^0 + M_{rr} \delta \varepsilon_{rr}^1 + N_{\theta\theta} \delta \varepsilon_{\theta\theta}^0 + M_{\theta\theta} \delta \varepsilon_{\theta\theta}^1] d\Omega_0 \quad (26)$$

Expanding out the strains in terms of the displacements, we have

$$\delta U = \int_{\Omega_0} \left[N_{rr} \frac{d\delta u_{0r}}{dr} - M_{rr} \frac{d^2\delta w_0}{dr^2} + \frac{N_{\theta\theta}}{r} \delta u_{0r} - \frac{M_{\theta\theta}}{r} \frac{d\delta w_0}{dr} \right] d\Omega_0 \quad (27)$$

Integration by parts leads to,

$$\begin{aligned} \delta U = & \oint_{\Gamma_0} n_r N_{rr} \delta u_{0r} d\Gamma_0 - \int_{\Omega_0} \frac{1}{r} \frac{d}{dr} (r N_{rr}) \delta u_{0r} d\Omega_0 \\ & - \oint_{\Gamma_0} n_r M_{rr} \frac{d\delta w_0}{dr} d\Gamma_0 + \int_{\Omega_0} \frac{1}{r} \frac{d}{dr} (r M_{rr}) \frac{d\delta w_0}{dr} d\Omega_0 + \int_{\Omega_0} \frac{N_{\theta\theta}}{r} \delta u_{0r} d\Omega_0 \\ & - \oint_{\Gamma_0} n_r \frac{M_{\theta\theta}}{r} \delta w_0 d\Gamma_0 + \int_{\Omega_0} \frac{1}{r} \frac{dM_{\theta\theta}}{dr} \delta w_0 d\Omega_0 \end{aligned} \quad (28)$$

keeping in mind that

$$\oint_{\Gamma_0} (\bullet) d\Gamma_0 = \int_{\theta} \left[(\bullet) \right]_{r_a}^{r_b} r d\theta ; \quad \int_{\Omega_0} (\bullet) d\Omega_0 = \int_{\theta} \int_r (\bullet) r dr d\theta \quad (29)$$

Let us define

$$\beta := \frac{dw_0}{dr} . \quad (30)$$

Then

$$\begin{aligned} \delta U = & \oint_{\Gamma_0} n_r \left(N_{rr} \delta u_{0r} - M_{rr} \delta \beta - \frac{M_{\theta\theta}}{r} \delta w_0 \right) d\Gamma_0 \\ & - \int_{\Omega_0} \frac{1}{r} \left[\left(\frac{d}{dr} (r N_{rr}) - N_{\theta\theta} \right) \delta u_{0r} - \frac{d}{dr} (r M_{rr}) \delta \beta - \frac{dM_{\theta\theta}}{dr} \delta w_0 \right] d\Omega_0 \end{aligned} \quad (31)$$

To remove the derivative of w_0 inside the area integral we integrate again by parts to get

$$\begin{aligned} \delta U = & \oint_{\Gamma_0} n_r \left[N_{rr} \delta u_{0r} - M_{rr} \delta \beta + \frac{1}{r} \left(\frac{d}{dr} (r M_{rr}) - M_{\theta\theta} \right) \delta w_0 \right] d\Gamma_0 \\ & - \int_{\Omega_0} \frac{1}{r} \left[\left(\frac{d}{dr} (r N_{rr}) - N_{\theta\theta} \right) \delta u_{0r} + \left(\frac{d^2}{dr^2} (r M_{rr}) - \frac{dM_{\theta\theta}}{dr} \right) \delta w_0 \right] d\Omega_0 \end{aligned} \quad (32)$$

The variation in the work done by the external forces is

$$\begin{aligned} \delta V_{\text{ext}} = & \int_{\Omega_0} [q(r) \delta w_0 + s(r) (\delta u_{0r} - z_f \delta \beta) + p(r) \delta u_{0\theta}] d\Omega_0 \\ & + \oint_{\Gamma_0} \int_{-f}^f [t_r (\delta u_{0r} - z \delta \beta) + t_\theta \delta u_{0\theta} + t_z \delta w_0] dz d\Gamma_0 \end{aligned} \quad (33)$$

where $q(r) = q^{\text{Top Face}}(r) + q^{\text{Bot Face}}(r)$ is a distributed surface force (per unit area) acting the positive z direction, $p(r) = p^{\text{Top Face}}(r) + p^{\text{Bot Face}}(r)$ is a distributed surface force (per unit area) acting the positive r direction, $s(r) = s^{\text{Top Face}}(r) + s^{\text{Bot Face}}(r)$ is a distributed surface force (per unit area) acting the positive θ direction, z_f takes the value $+f$ at the top of the facesheet and $-f$ at the bottom of the facesheet, and $\mathbf{t} = t_r \mathbf{e}_r + t_\theta \mathbf{e}_\theta + t_z \mathbf{e}_z$ is the surface traction vector.

A schematic of the loads there are applied to the facesheet is shown in Figure 2.

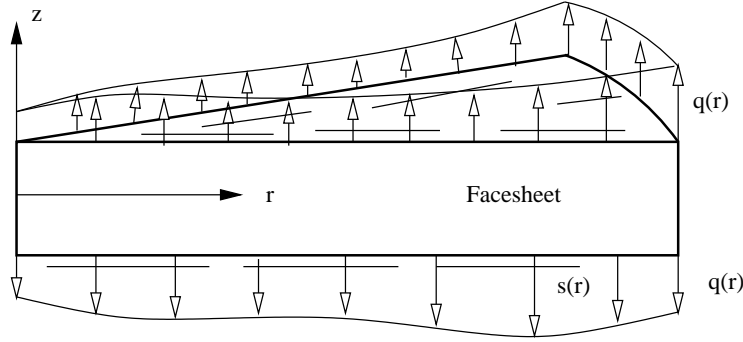


Figure 2 – The loads on a facesheet.

In terms of resultants over the thickness of the plate

$$\begin{aligned} \delta V_{\text{ext}} = & \int_{\Omega_0} [q(r) \delta w_0 + s(r) \delta u_{0r} - z_f s(r) \delta \beta + p(r) \delta u_{0\theta}] d\Omega_0 \\ & + \oint_{\Gamma_0} [N_r \delta u_{0r} - M_r \delta \beta + N_\theta \delta u_{0\theta} + Q_z \delta w_0] d\Gamma_0 \end{aligned} \quad (34)$$

where

$$N_r := \int_{-f}^f t_r dz ; \quad N_\theta := \int_{-f}^f t_\theta dz ; \quad Q_z := \int_{-f}^f t_z dz ; \quad M_r := \int_{-f}^f z t_r dz \quad (35)$$

Integrating the $\delta \beta$ term by parts over the area Ω_0 gives

$$\begin{aligned} \delta V_{\text{ext}} = & \int_{\Omega_0} \left[\left\{ q(r) + \frac{z_f}{r} \frac{d}{dr} (rs) \right\} \delta w_0 + s(r) \delta u_{0r} + p(r) \delta u_{0\theta} \right] d\Omega_0 \\ & + \oint_{\Gamma_0} [N_r \delta u_{0r} - M_r \delta \beta + N_\theta \delta u_{0\theta} + \{Q_z - n_r z_f s(r)\} \delta w_0] d\Gamma_0 \end{aligned} \quad (36)$$

Then, from the principle of virtual work, we have

$$\begin{aligned}
0 = & \oint_{\Gamma_0} \left[(n_r N_{rr} - N_r) \delta u_{0r} - (n_r M_{rr} - M_r) \delta \beta - N_\theta \delta u_{0\theta} \right. \\
& \left. + \left\{ \frac{n_r}{r} \left(\frac{d}{dr}(r M_{rr}) - M_{\theta\theta} \right) - Q_z + n_r z_f s(r) \right\} \delta w_0 \right] d\Gamma_0 \\
& - \int_{\Omega_0} \left[\left(\frac{1}{r} \frac{d}{dr}(r N_{rr}) - \frac{N_{\theta\theta}}{r} + s(r) \right) \delta u_{0r} + p(r) \delta u_{0\theta} \right. \\
& \left. + \left(\frac{1}{r} \frac{d^2}{dr^2}(r M_{rr}) - \frac{1}{r} \frac{dM_{\theta\theta}}{dr} + q(r) + \frac{z_f}{r} \frac{d}{dr}(rs) \right) \delta w_0 \right] d\Omega_0
\end{aligned} \tag{37}$$

Because of the arbitrariness of the virtual displacements, we have

$$\begin{aligned}
\int_{\Omega_0} \left(\frac{1}{r} \frac{d}{dr}(r N_{rr}) - \frac{N_{\theta\theta}}{r} + s(r) \right) \delta u_{0r} d\Omega_0 &= \oint_{\Gamma_0} (n_r N_{rr} - N_r) \delta u_{0r} d\Gamma_0 \\
\int_{\Omega_0} p(r) \delta u_{0\theta} &= - \oint_{\Gamma_0} N_\theta \delta u_{0\theta} d\Gamma_0 \\
\int_{\Omega_0} \left(\frac{1}{r} \frac{d^2}{dr^2}(r M_{rr}) - \frac{1}{r} \frac{dM_{\theta\theta}}{dr} + q(r) + \frac{z_f}{r} \frac{d}{dr}(rs) \right) \delta w_0 &= \oint_{\Gamma_0} \left[\left\{ \frac{n_r}{r} \left(\frac{d}{dr}(r M_{rr}) - M_{\theta\theta} \right) \right. \right. \\
&\quad \left. \left. - Q_z + n_r z_f s(r) \right\} \delta w_0 \right. \\
&\quad \left. - (n_r M_{rr} - M_r) \delta \beta \right] d\Gamma_0
\end{aligned} \tag{38}$$

Invoking the fundamental lemma of the calculus of variations (and keeping in mind that the displacement variations and the applied tractions are zero at points on the boundary where displacements are specified), we get the governing equations for the axisymmetric plate:

$$\begin{aligned}
\frac{1}{r} \frac{d}{dr}(r N_{rr}) - \frac{N_{\theta\theta}}{r} + s(r) &= 0 \\
p(r) &= 0 \\
\frac{1}{r} \frac{d^2}{dr^2}(r M_{rr}) - \frac{1}{r} \frac{dM_{\theta\theta}}{dr} + q(r) + \frac{z_f}{r} \frac{d}{dr}(rs) &= 0
\end{aligned} \tag{39}$$

Then the boundary conditions are

$$\begin{aligned}
\delta u_{0r} : \quad N_r &= n_r N_{rr} \\
\delta u_{0\theta} : \quad N_\theta &= 0 \\
\delta w_0 : \quad Q_z &= \frac{n_r}{r} \left[\frac{d}{dr}(r M_{rr}) - M_{\theta\theta} + z_f r s(r) \right] \\
\delta \beta : \quad M_r &= n_r M_{rr}
\end{aligned} \tag{40}$$

The governing equations are of order 6 in the displacements (u_{0r}, w_0) and there are 6 nontrivial boundary conditions, $(u_{0r}, w_0, \partial w_0 / \partial r, N_r, Q_z, M_r)$.

2.1.4 Summary of facesheet governing equations

The governing equations for the plate can then be summarized as follows:

- Equilibrium equations:

$$\begin{aligned} \frac{1}{r} \left[\frac{d}{dr} (r N_{rr}) - N_{\theta\theta} \right] + s(r) &= 0 \\ \frac{1}{r} \left[\frac{d^2}{dr^2} (r M_{rr}) - \frac{dM_{\theta\theta}}{dr} + z_f \frac{d}{dr} (r s) \right] + q(r) &= 0 \end{aligned} \quad (41)$$

- Stress-strain relations:

$$\begin{bmatrix} N_{rr} \\ N_{\theta\theta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{11} \end{bmatrix} \begin{bmatrix} \varepsilon_{rr}^0 \\ \varepsilon_{\theta\theta}^0 \end{bmatrix} \quad (42)$$

$$\begin{bmatrix} M_{rr} \\ M_{\theta\theta} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{11} \end{bmatrix} \begin{bmatrix} \varepsilon_{rr}^1 \\ \varepsilon_{\theta\theta}^1 \end{bmatrix} \quad (43)$$

- Strain-displacement relations:

$$\begin{aligned} \varepsilon_{rr}^0(r) &:= \frac{du_{0r}}{dr} ; \quad \varepsilon_{rr}^1(r) := -\frac{d^2 w_0}{dr^2} \\ \varepsilon_{\theta\theta}^0(r) &:= \frac{u_{0r}}{r} ; \quad \varepsilon_{\theta\theta}^1(r) := -\frac{1}{r} \frac{dw_0}{dr} \end{aligned} \quad (44)$$

- Boundary conditions:

$$\begin{aligned} \delta u_{0r} : \quad N_r &= n_r r N_{rr} \\ \delta w_0 : \quad Q_z &= \frac{n_r}{r} \left[\frac{d}{dr} (r M_{rr}) - M_{\theta\theta} + z_f r s(r) \right] \\ \delta \beta : \quad M_r &= n_r r M_{rr} \end{aligned} \quad (45)$$

2.1.5 Conversion into first-order ODEs

We would like to convert the governing equations for the axisymmetric plate into ODEs of first order for computational purposes. To do that, we note that the stress resultants are related to the displacements by

$$\begin{aligned} N_{rr} &= A_{11} \frac{du_{0r}}{dr} + A_{12} \frac{u_{0r}}{r} \\ N_{\theta\theta} &= A_{12} \frac{du_{0r}}{dr} + A_{11} \frac{u_{0r}}{r} \end{aligned} \quad (46)$$

From the first equation in (46), we have

$$\frac{du_{0r}}{dr} = \frac{N_{rr}}{A_{11}} - \frac{A_{12}}{A_{11}} \frac{u_{0r}}{r}. \quad (47)$$

Plugging the expression for $N_{\theta\theta}$ into the equilibrium equation for the stress resultants (41), we have

$$\frac{1}{r} \frac{d}{dr} (r N_{rr}) - \frac{A_{12}}{r} \frac{du_{0r}}{dr} - A_{11} \frac{u_{0r}}{r^2} + s(r) = 0 \quad (48)$$

Using (47),

$$\frac{dN_{rr}}{dr} + \left[1 - \frac{A_{12}}{A_{11}}\right] \frac{N_{rr}}{r} + \left(\frac{A_{12}^2}{A_{11}} - A_{11}\right) \frac{u_{0r}}{r^2} + s(r) = 0. \quad (49)$$

Recall that

$$\frac{dw_0}{dr} = \beta \quad (50)$$

Then the relations between the stress couples and the displacements take the form

$$\begin{aligned} M_{rr} &= -D_{11} \frac{d\beta}{dr} - D_{12} \frac{\beta}{r} \\ M_{\theta\theta} &= -D_{12} \frac{d\beta}{dr} - D_{11} \frac{\beta}{r} \end{aligned} \quad (51)$$

The first equation from (51) can be written as

$$\frac{d\beta}{dr} = -\frac{M_{rr}}{D_{11}} - \frac{D_{12}}{D_{11}} \frac{\beta}{r} \quad (52)$$

To convert the equilibrium equation for the stress couples into first-order ODEs, we define

$$Q_r := \frac{1}{r} \frac{d}{dr} (r M_{rr}) - \frac{M_{\theta\theta}}{r} + z_f s(r) \quad (53)$$

Then,

$$\frac{dM_{rr}}{dr} = Q_r + \frac{(M_{\theta\theta} - M_{rr})}{r} - z_f s(r) \quad (54)$$

Plugging in the expression for $M_{\theta\theta}$ from (51) and the expression for the derivative of β (52) we have

$$\frac{dM_{rr}}{dr} = Q_r + \left(\frac{D_{12} - D_{11}}{D_{11}}\right) \frac{M_{rr}}{r} + \left(\frac{D_{12}^2 - D_{11}^2}{D_{11}}\right) \frac{\beta}{r^2} - z_f s(r) \quad (55)$$

To reduce the order of the equilibrium equation for the stress couples, (41), we note that taking the derivative of Q_r from (53) gives us

$$r \frac{dQ_r}{dr} + Q_r = \frac{d^2}{dr^2} (r M_{rr}) - \frac{dM_{\theta\theta}}{dr} + z_f \frac{d}{dr} (rs) \quad (56)$$

Therefore the equilibrium equation for the stress couples can be written as

$$\frac{dQ_r}{dr} + \frac{Q_r}{r} + q(r) = 0 \quad (57)$$

2.1.6 Summary first-order ODEs for facesheets

The ODEs governing the facesheets are:

$$\frac{du_{0r}}{dr} = \frac{N_{rr}}{A_{11}} - \frac{A_{12}}{A_{11}} \frac{u_{0r}}{r} \quad (58)$$

$$\frac{dw_0}{dr} = \beta \quad (59)$$

$$\frac{d\beta}{dr} = -\frac{M_{rr}}{D_{11}} - \frac{D_{12}}{D_{11}} \frac{\beta}{r} \quad (60)$$

$$\frac{dN_{rr}}{dr} = \left[\frac{A_{12} - A_{11}}{A_{11}} \right] \frac{N_{rr}}{r} + \left[\frac{A_{11}^2 - A_{12}^2}{A_{11}} \right] \frac{u_{0r}}{r^2} - s(r) \quad (61)$$

$$\frac{dM_{rr}}{dr} = Q_r + \left[\frac{D_{12} - D_{11}}{D_{11}} \right] \frac{M_{rr}}{r} + \left[\frac{D_{12}^2 - D_{11}^2}{D_{11}} \right] \frac{\beta}{r^2} - z_f s(r) \quad (62)$$

$$\frac{dQ_r}{dr} = -\frac{Q_r}{r} - q(r) \quad (63)$$

and the boundary conditions are

$$\begin{aligned} u_{0r} : \quad N_r &= n_r r N_{rr} \\ w_0 : \quad Q_z &= n_r r Q_r \\ \beta : \quad M_r &= n_r r M_{rr} \end{aligned} \quad (64)$$

2.2 Core equations

2.2.1 Stress-strain relations

We assume that the core is transversely isotropic. In that case, the stress-strain relations in the core have the form

$$\begin{aligned} \sigma_{rr} &= C_{11} \varepsilon_{rr} + C_{12} \varepsilon_{\theta\theta} + C_{13} \varepsilon_{zz} \\ \sigma_{\theta\theta} &= C_{12} \varepsilon_{rr} + C_{11} \varepsilon_{\theta\theta} + C_{13} \varepsilon_{zz} \\ \sigma_{zz} &= C_{13} \varepsilon_{rr} + C_{13} \varepsilon_{\theta\theta} + C_{33} \varepsilon_{zz} \\ \sigma_{\theta z} &= 0 ; \quad \sigma_{rz} = C_{55} \varepsilon_{rz} ; \quad \sigma_{r\theta} = 0 \end{aligned} \quad (65)$$

If we also assume that the core cannot sustain any in-plane stresses, then

$$\begin{aligned} \sigma_{rr} = 0 &= C_{11} \varepsilon_{rr} + C_{12} \varepsilon_{\theta\theta} + C_{13} \varepsilon_{zz} \\ \sigma_{\theta\theta} = 0 &= C_{12} \varepsilon_{rr} + C_{11} \varepsilon_{\theta\theta} + C_{13} \varepsilon_{zz} \end{aligned} \quad (66)$$

Therefore we have

$$(C_{11} - C_{12}) (\varepsilon_{rr} - \varepsilon_{\theta\theta}) = 0 \quad (67)$$

which implies that $C_{11} = C_{12}$. If we assume that $C_{11} = C_{12} = \epsilon C_{13}$ where $\epsilon \ll 1$ is a positive quantity, then we have $C_{13} = 0$. Therefore the stress-strain relations in the core reduce to

$$\sigma_{rr} = 0 ; \quad \sigma_{\theta\theta} = 0 ; \quad \sigma_{zz} = C_{33} \varepsilon_{zz} ; \quad \sigma_{\theta z} = 0 ; \quad \sigma_{rz} = C_{55} \varepsilon_{rz} ; \quad \sigma_{r\theta} = 0 \quad (68)$$

2.2.2 Strain-displacement relations

From the strain-displacement relations we have

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} ; \quad \varepsilon_{rz} = \frac{1}{2} \left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right] \quad (69)$$

2.2.3 Stress-displacement relations

Using the stress-strain relations we get

$$\sigma_{zz} = C_{33} \frac{\partial u_z}{\partial z} ; \quad \sigma_{rz} = \frac{C_{55}}{2} \left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right] \quad (70)$$

2.2.4 Equilibrium equations

The equilibrium equations also reduce accordingly to

$$\frac{\partial \sigma_{rz}}{\partial z} = 0 ; \quad \frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}}{r} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \quad (71)$$

2.2.5 Expression for u_z

Recall

$$\sigma_{zz} = C_{33} \frac{\partial u_z}{\partial z} \quad \implies \quad \frac{\partial u_z}{\partial z} = S_{33} \sigma_{zz} \quad (72)$$

where $S_{33} := 1/C_{33}$.

Integrating, we get

$$u_z(r, z) = \int_{z_a}^{z_b} S_{33} \sigma_{zz} dz + A(r) \quad (73)$$

where $A(r)$ is a function only of r . Integrating by parts, we have

$$u_z(r, z) = S_{33} \left[z \sigma_{zz} \Big|_{z_a}^{z_b} - \int_{z_a}^{z_b} z \frac{\partial \sigma_{zz}}{\partial z} dz \right] + A(r) \quad (74)$$

Now we assume that the displacement u_z is quadratic in z to get

$$\frac{\partial \sigma_{zz}}{\partial z} = -\frac{\partial \sigma_{rz}}{\partial r} - \frac{\sigma_{rz}}{r} =: B(r) \quad (75)$$

where $B(r)$ is a function only of r . If we set up the coordinate system in the core such that $z_c = z - c$ where $2c$ is the core thickness and integrate from 0 to z_c , we get

$$\begin{aligned} u_z(r, z_c) &= S_{33} \left[z_c \sigma_{zz}(r, z_c) - B(r) \int_{z_c}^0 z dz \right] + A(r) \\ &= S_{33} \left[z_c \sigma_{zz}(r, z_c) - B(r) \frac{z_c^2}{2} \right] + A(r) \end{aligned} \quad (76)$$

At $z_c = c$ the displacement of the core is equal to the displacement of the top facesheet, i.e.,

$$w^1(r) = u_z(r, c) = S_{33} \left[c \sigma_{zz}(r, c) - B(r) \frac{c^2}{2} \right] + A(r) \quad (77)$$

Eliminating $A(r)$, we get

$$u_z(r, z_c) = w^1(r) + S_{33} \left[\{ z_c \sigma_{zz}(r, z_c) - c \sigma_{zz}(r, c) \} - \frac{B(r)}{2} (z_c^2 - c^2) \right]. \quad (78)$$

We can also calculate the displacement at the bottom facesheet

$$w^2(r) = u_z(r, -c) = S_{33} \left[-c \sigma_{zz}(r, -c) - B(r) \frac{c^2}{2} \right] + A(r) \quad (79)$$

Again, eliminating $A(r)$, we have

$$w^1(r) - w^2(r) = c S_{33} [\sigma_{zz}(r, c) + \sigma_{zz}(r, -c)]. \quad (80)$$

2.2.6 Eliminating σ_{zz}

We would like to eliminate σ_{zz} from the expression in equation (90). To do that, we recall that

$$\frac{\partial \sigma_{zz}}{\partial z} = B(r) \quad (81)$$

Integrating between the limits 0 and z_c as before, we get

$$\sigma_{zz}(r, z_c) = B(r) z_c + E(r) \quad (82)$$

where $E(r)$ is a function of r only. Therefore,

$$\sigma_{zz}(r, c) = B(r) c + E(r); \quad \sigma_{zz}(r, -c) = -B(r) c + E(r) \quad (83)$$

which gives

$$E(r) = \sigma_{zz}(r, c) - c B(r). \quad (84)$$

Therefore,

$$\sigma_{zz}(r, z_c) = (z_c - c) B(r) + \sigma_{zz}(r, c) \quad (85)$$

We also have,

$$\sigma_{zz}(r, c) + \sigma_{zz}(r, -c) = 2E(r) = 2[\sigma_{zz}(r, c) - c B(r)] \quad (86)$$

Hence, from (80),

$$w^1(r) - w^2(r) = 2 c S_{33} [\sigma_{zz}(r, c) - c B(r)] \quad (87)$$

or,

$$\sigma_{zz}(r, c) = \frac{C_{33}}{2c} [w^1(r) - w^2(r)] + c B(r) \quad (88)$$

Combining (85) and (88),

$$\sigma_{zz}(r, z_c) = \frac{C_{33}}{2c} [w^1(r) - w^2(r)] + z_c B(r) \quad (89)$$

Using (88) and (89) in (78) gives

$$u_z(r, z_c) = w^1(r) + \left(\frac{z_c - c}{2} \right) \left[\frac{w^1(r) - w^2(r)}{c} + (z_c + c) S_{33} B(r) \right]. \quad (90)$$

Now, from equations (10) for the facesheets, we have

$$w^1(r) = w_0^{\text{top}}; \quad w^2(r) = w_0^{\text{bot}} \quad (91)$$

respectively. Plugging these into (90) gives

$$u_z(r, z_c) = \frac{1}{2} \left(\frac{z_c}{c} + 1 \right) w_0^{\text{top}} - \frac{1}{2} \left(\frac{z_c}{c} - 1 \right) w_0^{\text{bot}} + \frac{1}{2} (z_c^2 - c^2) S_{33} B(r). \quad (92)$$

2.2.7 Expression for u_r

Recall that

$$\sigma_{rz} = \frac{C_{55}}{2} \left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right] \quad (93)$$

Therefore,

$$\frac{\partial u_r}{\partial z} = 2 S_{55} \sigma_{rz} - \frac{\partial u_z}{\partial r}; \quad S_{55} := 1/C_{55} \quad (94)$$

Also, taking the r -derivative of equation (92), we have

$$\frac{\partial u_z}{\partial r} = \frac{1}{2} \left(\frac{z_c}{c} + 1 \right) \frac{\partial w_0^{\text{top}}}{\partial r} - \frac{1}{2} \left(\frac{z_c}{c} - 1 \right) \frac{\partial w_0^{\text{bot}}}{\partial r} + \frac{1}{2} (z_c^2 - c^2) S_{33} \frac{\partial B}{\partial r}. \quad (95)$$

Substitution of (95) into (94) gives

$$\frac{\partial u_r}{\partial z} = 2 S_{55} \sigma_{rz} - \frac{1}{2} \left(\frac{z_c}{c} + 1 \right) \frac{dw_0^{\text{top}}}{dr} + \frac{1}{2} \left(\frac{z_c}{c} - 1 \right) \frac{dw_0^{\text{bot}}}{dr} - \frac{1}{2} (z_c^2 - c^2) S_{33} \frac{dB}{dr}. \quad (96)$$

Note that $\frac{\partial \sigma_{rz}}{\partial z} = 0 \implies \sigma_{rz} = \sigma_{rz}(r)$. Integrating (96) between 0 and z_c , we get

$$\begin{aligned} u_r(r, z_c) = 2 S_{55} z_c \sigma_{rz} - \frac{1}{2} \left(\frac{z_c^2}{2c} + z_c \right) \frac{dw_0^{\text{top}}}{dr} + \frac{1}{2} \left(\frac{z_c^2}{2c} - z_c \right) \frac{dw_0^{\text{bot}}}{dr} \\ - \frac{1}{2} \left(\frac{z_c^3}{3} - c^2 z_c \right) S_{33} \frac{dB}{dr} + G(r) \end{aligned} \quad (97)$$

where $G(r)$ is a function only of r .

At $z_c = c$, $u_r = u^1(r)$. Hence we have

$$G(r) = u^1 - 2 S_{55} c \sigma_{rz} + \frac{3c}{4} \frac{dw_0^{\text{top}}}{dr} + \frac{c}{4} \frac{dw_0^{\text{bot}}}{dr} - \frac{c^3}{3} S_{33} \frac{dB}{dr} \quad (98)$$

Substitution of (98) into (97) gives

$$u_r(r, z_c) = u^1 + 2 S_{55} (z_c - c) \sigma_{rz} + \left[\frac{3c}{4} - \frac{1}{2} \left(\frac{z_c^2}{2c} + z_c \right) \right] \frac{dw_0^{\text{top}}}{dr} + \left[\frac{c}{4} + \frac{1}{2} \left(\frac{z_c^2}{2c} - z_c \right) \right] \frac{dw_0^{\text{bot}}}{dr} - \left[\frac{c^3}{3} + \frac{1}{2} \left(\frac{z_c^3}{3} - c^2 z_c \right) \right] S_{33} \frac{dB}{dr} \quad (99)$$

Now, from equations (10) and (30) for the facesheets, we have

$$\begin{aligned} \frac{dw_0^{\text{top}}}{dr} &=: \beta^{\text{top}} ; \quad \frac{dw_0^{\text{bot}}}{dr} =: \beta^{\text{bot}} \\ u^1(r) &= u_{0r}^{\text{top}} + f^{\text{top}} \beta^{\text{top}} ; \quad u^2(r) = u_{0r}^{\text{bot}} - f^{\text{bot}} \beta^{\text{bot}} \end{aligned} \quad (100)$$

where $2 f^{\text{top}}$ and $2 f^{\text{bot}}$ are the thicknesses of the top and bottom facesheets, respectively. Plugging these into (99) gives

$$\begin{aligned} u_r(r, z_c) &= u_{0r}^{\text{top}} + f^{\text{top}} \beta^{\text{top}} + 2 S_{55} (z_c - c) \sigma_{rz} + \left[\frac{3c}{4} - \frac{1}{2} \left(\frac{z_c^2}{2c} + z_c \right) \right] \beta^{\text{top}} \\ &\quad + \left[\frac{c}{4} + \frac{1}{2} \left(\frac{z_c^2}{2c} - z_c \right) \right] \beta^{\text{bot}} - \left[\frac{c^3}{3} + \frac{1}{2} \left(\frac{z_c^3}{3} - c^2 z_c \right) \right] S_{33} \frac{dB}{dr} \end{aligned} \quad (101)$$

or,

$$\begin{aligned} u_r(r, z_c) &= u_{0r}^{\text{top}} + \left[f^{\text{top}} + \frac{3c}{4} - \frac{1}{2} \left(\frac{z_c^2}{2c} + z_c \right) \right] \beta^{\text{top}} + \left[\frac{c}{4} + \frac{1}{2} \left(\frac{z_c^2}{2c} - z_c \right) \right] \beta^{\text{bot}} \\ &\quad + 2 S_{55} (z_c - c) \sigma_{rz} - \left[\frac{c^3}{3} + \frac{1}{2} \left(\frac{z_c^3}{3} - c^2 z_c \right) \right] S_{33} \frac{dB}{dr} \end{aligned} \quad (102)$$

2.2.8 Governing equation for the core

Now, at the bottom of the core, $z_c = -c$. From (102) we have

$$u_r(r, -c) = u^2 = u_{0r}^{\text{top}} + [f^{\text{top}} + c] \beta^{\text{top}} + c \beta^{\text{bot}} - 4 S_{55} \sigma_{rz} c - \frac{2}{3} S_{33} c^3 \frac{dB(r)}{dr} \quad (103)$$

Also

$$u^2 = u_{0r}^{\text{bot}} - f^{\text{bot}} \beta^{\text{bot}}$$

Hence

$$0 = u_{0r}^{\text{top}} - u_{0r}^{\text{bot}} + [f^{\text{top}} + c] \beta^{\text{top}} + [f^{\text{bot}} + c] \beta^{\text{bot}} - 4 S_{55} \sigma_{rz} c - \frac{2}{3} S_{33} c^3 \frac{dB(r)}{dr} \quad (104)$$

Recall that

$$B = -\frac{d\sigma_{rz}}{dr} - \frac{\sigma_{rz}}{r}$$

Therefore,

$$\frac{dB}{dr} = -\frac{d^2\sigma_{rz}}{dr^2} + \frac{\sigma_{rz}}{r^2} - \frac{1}{r} \frac{d\sigma_{rz}}{dr} \quad (105)$$

Plugging (105) into (104) gives

$$0 = u_{0r}^{\text{top}} - u_{0r}^{\text{bot}} + [f^{\text{top}} + c] \beta^{\text{top}} + [f^{\text{bot}} + c] \beta^{\text{bot}} - 4 S_{55} \sigma_{rz} c - \frac{2}{3} S_{33} c^3 \left[-\frac{d^2 \sigma_{rz}}{dr^2} + \frac{\sigma_{rz}}{r^2} - \frac{1}{r} \frac{d\sigma_{rz}}{dr} \right] \quad (106)$$

or,

$$\begin{aligned} \frac{d^2 \sigma_{rz}}{dr^2} + \frac{1}{r} \frac{d\sigma_{rz}}{dr} - \left(\frac{1}{r^2} + \frac{6C_{33}S_{55}}{c^2} \right) \sigma_{rz} \\ = -\frac{3C_{33}}{2c^3} \left[u_{0r}^{\text{top}} - u_{0r}^{\text{bot}} + (f^{\text{top}} + c) \beta^{\text{top}} + (f^{\text{bot}} + c) \beta^{\text{bot}} \right] \end{aligned} \quad (107)$$

2.2.9 Conversion into first order ODEs

To convert (107) into first-order ODEs, we define

$$T_r := \frac{d\sigma_{rz}}{dr} \quad (108)$$

Then equation (107) can be written as

$$\begin{aligned} \frac{dT_r}{dr} = -\frac{3C_{33}}{2c^3} \left[u_{0r}^{\text{top}} - u_{0r}^{\text{bot}} + (f^{\text{top}} + c) \beta^{\text{top}} + (f^{\text{bot}} + c) \beta^{\text{bot}} \right] \\ + \left(\frac{1}{r^2} + \frac{6C_{33}S_{55}}{c^2} \right) \sigma_{rz} - \frac{T_r}{r} \end{aligned} \quad (109)$$

2.2.10 Summary of first order ODEs for the core

The governing equations for the stresses in the core are

$$\frac{d\sigma_{rz}}{dr} = T_r \quad (110)$$

$$\begin{aligned} \frac{dT_r}{dr} = -\frac{3C_{33}}{2c^3} \left[u_{0r}^{\text{top}} - u_{0r}^{\text{bot}} + (f^{\text{top}} + c) \beta^{\text{top}} + (f^{\text{bot}} + c) \beta^{\text{bot}} \right] \\ + \left(\frac{1}{r^2} + \frac{6C_{33}S_{55}}{c^2} \right) \sigma_{rz} - \frac{T_r}{r} \end{aligned} \quad (111)$$

$$\sigma_{zz}(r, z_c) = \frac{C_{33}}{2c} [w^1(r) - w^2(r)] - z_c T_r - z_c \frac{\sigma_{rz}}{r} \quad (112)$$

3 Coupled governing equations of the facesheets and the core

In the previous section, ODEs have been derived that partially couple the core to the facesheets. To complete the coupling of the facesheets to the core we have to balance the forces at the interfaces between the core and the facesheets. We introduce some new notation to aid us in the coupling process. Recall that for a facesheet

$$s(r) = s^{\text{Top Face}} + s^{\text{Bot Face}} ; \quad q(r) = q^{\text{Top Face}} + q^{\text{Bot Face}} \quad (113)$$

We identify these two sets of applied tractions on the two facesheets using the notation

$$s^{\text{top}}(r) = s^{\text{tt}} + s^{\text{tb}} ; \quad q^{\text{top}}(r) = q^{\text{tt}} + q^{\text{tb}} ; \quad s^{\text{bot}}(r) = s^{\text{bt}} + s^{\text{bb}} ; \quad q^{\text{bot}}(r) = q^{\text{bt}} + q^{\text{bb}} \quad (114)$$

The tractions at the core-facesheet interface are given by

$$\begin{aligned} \mathbf{t} &= t_r \mathbf{e}_r + t_\theta \mathbf{e}_\theta + t_z \mathbf{e}_z \\ &= (n_r \sigma_{rr} + n_\theta \sigma_{r\theta} + n_z \sigma_{rz}) \mathbf{e}_r + (n_r \sigma_{r\theta} + n_\theta \sigma_{\theta\theta} + n_z \sigma_{\theta z}) \mathbf{e}_\theta \\ &\quad + (n_r \sigma_{rz} + n_\theta \sigma_{\theta z} + n_z \sigma_{zz}) \mathbf{e}_z \end{aligned} \quad (115)$$

where $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ are the basis vectors in the r, θ, z directions. In the core $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{\theta z} = \sigma_{r\theta} = 0$. Therefore, the traction vector simplifies to

$$\mathbf{t} = n_z \sigma_{rz} \mathbf{e}_r + (n_r \sigma_{rz} + n_z \sigma_{zz}) \mathbf{e}_z \quad (116)$$

At the interface between the core and the top facesheet, $n_r = 0, n_z = 1$ while at the interface between the core and the bottom facesheet $n_r = 0, n_z = -1$. Therefore,

$$\mathbf{t}^{\text{tc}}(r) = \sigma_{rz}^{\text{core}}(r) \mathbf{e}_r + \sigma_{zz}^{\text{core}}(r, c) \mathbf{e}_z ; \quad \mathbf{t}^{\text{bc}}(r) = -\sigma_{rz}^{\text{core}}(r) \mathbf{e}_r - \sigma_{zz}^{\text{core}}(r, -c) \mathbf{e}_z \quad (117)$$

To couple the facesheet equations to the core equations we have, due to the continuity of tractions at the core-facesheet interfaces,

$$\begin{aligned} s^{\text{tb}}(r) + \mathbf{t}^{\text{tc}}(r) \cdot \mathbf{e}_r &= 0 \implies s^{\text{tb}}(r) = -\sigma_{rz}^{\text{core}}(r) \\ s^{\text{bt}}(r) + \mathbf{t}^{\text{bc}}(r) \cdot \mathbf{e}_r &= 0 \implies s^{\text{bt}}(r) = \sigma_{rz}^{\text{core}}(r) \\ q^{\text{tb}}(r) + \mathbf{t}^{\text{tc}}(r) \cdot \mathbf{e}_z &= 0 \implies q^{\text{tb}}(r) = -\sigma_{zz}^{\text{core}}(r, c) \\ q^{\text{bt}}(r) + \mathbf{t}^{\text{bc}}(r) \cdot \mathbf{e}_z &= 0 \implies q^{\text{bt}}(r) = \sigma_{zz}^{\text{core}}(r, -c) \end{aligned} \quad (118)$$

From equations (89) and (75)

$$\sigma_{zz}^{\text{core}}(r, z_c) = \frac{C_{33}}{2c} [w_0^{\text{top}} - w_0^{\text{bot}}] - z_c \frac{d\sigma_{rz}^{\text{core}}}{dr} - z_c \frac{\sigma_{rz}^{\text{core}}}{r} \quad (119)$$

Therefore,

$$\begin{aligned} q^{\text{tb}}(r) &= -\frac{C_{33}^{\text{core}}}{2c} [w_0^{\text{top}} - w_0^{\text{bot}}] + c \frac{d\sigma_{rz}^{\text{core}}}{dr} + c \frac{\sigma_{rz}^{\text{core}}}{r} \\ q^{\text{bt}}(r) &= \frac{C_{33}^{\text{core}}}{2c} [w_0^{\text{top}} - w_0^{\text{bot}}] + c \frac{d\sigma_{rz}^{\text{core}}}{dr} + c \frac{\sigma_{rz}^{\text{core}}}{r} \end{aligned} \quad (120)$$

Equation (57) then takes the form

$$\begin{aligned} \frac{dQ_r^{\text{top}}}{dr} &= -\frac{Q_r^{\text{top}}}{r} - q^{\text{top}}(r) = -\frac{Q_r^{\text{top}}}{r} + \frac{C_{33}^{\text{core}}}{2c} [w_0^{\text{top}} - w_0^{\text{bot}}] - c \frac{d\sigma_{rz}^{\text{core}}}{dr} - c \frac{\sigma_{rz}^{\text{core}}}{r} - q^{\text{tt}} \\ \frac{dQ_r^{\text{bot}}}{dr} &= -\frac{Q_r^{\text{bot}}}{r} - q^{\text{bot}}(r) = -\frac{Q_r^{\text{bot}}}{r} - \frac{C_{33}^{\text{core}}}{2c} [w_0^{\text{top}} - w_0^{\text{bot}}] - c \frac{d\sigma_{rz}^{\text{core}}}{dr} - c \frac{\sigma_{rz}^{\text{core}}}{r} - q^{\text{bb}} \end{aligned} \quad (121)$$

Similarly, equation (49) takes the form

$$\frac{dN_{rr}^{\text{top}}}{dr} = \left[\frac{A_{12}^{\text{top}} - A_{11}^{\text{top}}}{A_{11}^{\text{top}}} \right] \frac{N_{rr}^{\text{top}}}{r} + \left[\frac{(A_{11}^{\text{top}})^2 - (A_{12}^{\text{top}})^2}{A_{11}^{\text{top}}} \right] \frac{u_{0r}^{\text{top}}}{r^2} + \sigma_{rz}^{\text{core}} - s^{\text{tt}} \quad (122)$$

and

$$\frac{dN_{rr}^{\text{bot}}}{dr} = \left[\frac{A_{12}^{\text{bot}} - A_{11}^{\text{bot}}}{A_{11}^{\text{bot}}} \right] \frac{N_{rr}^{\text{bot}}}{r} + \left[\frac{(A_{11}^{\text{bot}})^2 - (A_{12}^{\text{bot}})^2}{A_{11}^{\text{bot}}} \right] \frac{u_{0r}^{\text{bot}}}{r^2} - \sigma_{rz}^{\text{core}} - s^{\text{bb}} \quad (123)$$

Also, equation (55) takes the form

$$\frac{dM_{rr}^{\text{top}}}{dr} = Q_r^{\text{top}} + \left[\frac{D_{12}^{\text{top}} - D_{11}^{\text{top}}}{D_{11}^{\text{top}}} \right] \frac{M_{rr}^{\text{top}}}{r} + \left[\frac{(D_{12}^{\text{top}})^2 - (D_{11}^{\text{top}})^2}{D_{11}^{\text{top}}} \right] \frac{\beta^{\text{top}}}{r^2} + f^{\text{top}} (s^{\text{tt}} - \sigma_{rz}^{\text{core}}) \quad (124)$$

and

$$\frac{dM_{rr}^{\text{bot}}}{dr} = Q_r^{\text{bot}} + \left[\frac{D_{12}^{\text{bot}} - D_{11}^{\text{bot}}}{D_{11}^{\text{bot}}} \right] \frac{M_{rr}^{\text{bot}}}{r} + \left[\frac{(D_{12}^{\text{bot}})^2 - (D_{11}^{\text{bot}})^2}{D_{11}^{\text{bot}}} \right] \frac{\beta^{\text{bot}}}{r^2} - f^{\text{bot}} (s^{\text{bb}} + \sigma_{rz}^{\text{core}}) \quad (125)$$

The governing first order ODEs for the facesheets and the core can then be expressed as

- **Top facesheet:**

$$\begin{aligned} \frac{du_{0r}^{\text{top}}}{dr} &= \frac{N_{rr}^{\text{top}}}{A_{11}^{\text{top}}} - \frac{A_{12}^{\text{top}}}{A_{11}^{\text{top}}} \frac{u_{0r}^{\text{top}}}{r} \\ \frac{dw_0^{\text{top}}}{dr} &= \beta^{\text{top}} \\ \frac{d\beta^{\text{top}}}{dr} &= -\frac{M_{rr}^{\text{top}}}{D_{11}^{\text{top}}} - \frac{D_{12}^{\text{top}}}{D_{11}^{\text{top}}} \frac{\beta^{\text{top}}}{r} \\ \frac{dN_{rr}^{\text{top}}}{dr} &= \left[\frac{A_{12}^{\text{top}} - A_{11}^{\text{top}}}{A_{11}^{\text{top}}} \right] \frac{N_{rr}^{\text{top}}}{r} + \left[\frac{(A_{11}^{\text{top}})^2 - (A_{12}^{\text{top}})^2}{A_{11}^{\text{top}}} \right] \frac{u_{0r}^{\text{top}}}{r^2} + \sigma_{rz}^{\text{core}} - s^{\text{tt}} \\ \frac{dM_{rr}^{\text{top}}}{dr} &= Q_r^{\text{top}} + \left[\frac{D_{12}^{\text{top}} - D_{11}^{\text{top}}}{D_{11}^{\text{top}}} \right] \frac{M_{rr}^{\text{top}}}{r} + \left[\frac{(D_{12}^{\text{top}})^2 - (D_{11}^{\text{top}})^2}{D_{11}^{\text{top}}} \right] \frac{\beta^{\text{top}}}{r^2} + f^{\text{top}} (s^{\text{tt}} - \sigma_{rz}^{\text{core}}) \\ \frac{dQ_r^{\text{top}}}{dr} &= -\frac{Q_r^{\text{top}}}{r} + \frac{C_{33}^{\text{core}}}{2c} [w_0^{\text{top}}(r) - w_0^{\text{bot}}(r)] - c T_r^{\text{core}} - c \frac{\sigma_{rz}^{\text{core}}}{r} - q^{\text{tt}} \end{aligned} \quad (126)$$

• **Bottom facesheet:**

$$\begin{aligned}
 \frac{du_{0r}^{\text{bot}}}{dr} &= \frac{N_{rr}^{\text{bot}}}{A_{11}^{\text{bot}}} - \frac{A_{12}^{\text{bot}}}{A_{11}^{\text{bot}}} \frac{u_{0r}^{\text{bot}}}{r} \\
 \frac{dw_0^{\text{bot}}}{dr} &= \beta^{\text{bot}} \\
 \frac{d\beta^{\text{bot}}}{dr} &= -\frac{M_{rr}^{\text{bot}}}{D_{11}^{\text{bot}}} - \frac{D_{12}^{\text{bot}}}{D_{11}^{\text{bot}}} \frac{\beta^{\text{bot}}}{r} \\
 \frac{dN_{rr}^{\text{bot}}}{dr} &= \left[\frac{A_{12}^{\text{bot}} - A_{11}^{\text{bot}}}{A_{11}^{\text{bot}}} \right] \frac{N_{rr}^{\text{bot}}}{r} + \left[\frac{(A_{11}^{\text{bot}})^2 - (A_{12}^{\text{bot}})^2}{A_{11}^{\text{bot}}} \right] \frac{u_{0r}^{\text{bot}}}{r^2} - \sigma_{rz}^{\text{core}} - s^{\text{bb}} \\
 \frac{dM_{rr}^{\text{bot}}}{dr} &= Q_r^{\text{bot}} + \left[\frac{D_{12}^{\text{bot}} - D_{11}^{\text{bot}}}{D_{11}^{\text{bot}}} \right] \frac{M_{rr}^{\text{bot}}}{r} + \left[\frac{(D_{12}^{\text{bot}})^2 - (D_{11}^{\text{bot}})^2}{D_{11}^{\text{bot}}} \right] \frac{\beta^{\text{bot}}}{r^2} - f^{\text{bot}} (s^{\text{bb}} + \sigma_{rz}^{\text{core}}) \\
 \frac{dQ_r^{\text{bot}}}{dr} &= -\frac{Q_r^{\text{bot}}}{r} - \frac{C_{33}^{\text{core}}}{2c} \left[w_0^{\text{top}}(r) - w_0^{\text{bot}}(r) \right] - c T_r^{\text{core}} - c \frac{\sigma_{rz}^{\text{core}}}{r} - q^{\text{bb}}
 \end{aligned} \tag{127}$$

• **Core:**

$$\begin{aligned}
 \frac{d\sigma_{rz}^{\text{core}}}{dr} &= T_r^{\text{core}} \\
 \frac{dT_r^{\text{core}}}{dr} &= -\frac{3C_{33}^{\text{core}}}{2c^3} \left[u_{0r}^{\text{top}} - u_{0r}^{\text{bot}} + (f^{\text{top}} + c) \beta^{\text{top}} + (f^{\text{bot}} + c) \beta^{\text{bot}} \right] \\
 &\quad + \left(\frac{1}{r^2} + \frac{6C_{33}^{\text{core}} S_{55}^{\text{core}}}{c^2} \right) \sigma_{rz}^{\text{core}} - \frac{T_r^{\text{core}}}{r}
 \end{aligned} \tag{128}$$

This is a set of 14 coupled ODEs that can be solved using a number of approaches. Thomsen and coworkers [7, 8] use a multi-segment integration approach to solve these equations. Since it is considerably simple to solve the original system of equations using the finite element approach, we have used finite elements in this work.

4 Finite element formulation of the coupled governing equations

For the finite element formulation of the governing equations, it is convenient to start with the statement of virtual work for the facesheets, i.e.,

$$\begin{aligned}
 \int_{\Omega_0} \left[N_{rr} \frac{d\delta u_{0r}}{dr} + \frac{N_{\theta\theta}}{r} \delta u_{0r} - M_{rr} \frac{d^2 \delta w_0}{dr^2} - \frac{M_{\theta\theta}}{r} \frac{d\delta w_0}{dr} \right] d\Omega_0 = \\
 \int_{\Omega_0} \left[s(r) \delta u_{0r} - z_f s(r) \frac{d\delta w_0}{dr} + q(r) \delta w_0 \right] d\Omega_0 \\
 + \oint_{\Gamma_0} \left[N_r \delta u_{0r} - M_r \frac{d\delta w_0}{dr} + Q_z \delta w_0 \right] d\Gamma_0
 \end{aligned} \tag{129}$$

where

$$\begin{aligned}
 N_{rr} &= A_{11} \frac{du_{0r}}{dr} + A_{12} \frac{u_{0r}}{r}; \quad N_{\theta\theta} = A_{12} \frac{du_{0r}}{dr} + A_{11} \frac{u_{0r}}{r} \\
 M_{rr} &= -D_{11} \frac{d^2 w_0}{dr^2} - \frac{D_{12}}{r} \frac{dw_0}{dr}; \quad M_{\theta\theta} = -D_{12} \frac{d^2 w_0}{dr^2} - \frac{D_{11}}{r} \frac{dw_0}{dr}
 \end{aligned} \tag{130}$$

Separating terms containing δu_{0r} and δw_0 leads to two equations

$$\begin{aligned} \int_{\Omega_0} \left[N_{rr} \frac{d\delta u_{0r}}{dr} + \left\{ \frac{N_{\theta\theta}}{r} - s(r) \right\} \delta u_{0r} \right] d\Omega_0 &= \oint_{\Gamma_0} N_r \delta u_{0r} d\Gamma_0 \quad (131) \\ \int_{\Omega_0} \left[M_{rr} \frac{d^2\delta w_0}{dr^2} + \left\{ \frac{M_{\theta\theta}}{r} + z_f s(r) \right\} \frac{d\delta w_0}{dr} + q(r) \delta w_0 \right] d\Omega_0 &= \oint_{\Gamma_0} \left[M_r \frac{d\delta w_0}{dr} - Q_z \delta w_0 \right] d\Gamma_0 \quad (132) \end{aligned}$$

The continuity of tractions across the facesheet-core interfaces requires that

$$\begin{aligned} q^{\text{top}}(r) &= q^{\text{tb}}(r) + q^{\text{tt}}(r) = -\frac{C_{33}^{\text{core}}}{2c} [w_0^{\text{top}}(r) - w_0^{\text{bot}}(r)] + c \frac{d\sigma_{rz}^{\text{core}}}{dr} + c \frac{\sigma_{rz}^{\text{core}}}{r} + q^{\text{tt}}(r) \quad (133) \\ q^{\text{bot}}(r) &= q^{\text{bt}}(r) + q^{\text{bb}}(r) = \frac{C_{33}^{\text{core}}}{2c} [w_0^{\text{top}}(r) - w_0^{\text{bot}}(r)] + c \frac{d\sigma_{rz}^{\text{core}}}{dr} + c \frac{\sigma_{rz}^{\text{core}}}{r} + q^{\text{bb}}(r) \end{aligned}$$

and

$$\begin{aligned} s^{\text{top}}(r) &= s^{\text{tb}}(r) + s^{\text{tt}}(r) = -\sigma_{rz}^{\text{core}}(r) + s^{\text{tt}}(r) \quad (134) \\ s^{\text{bot}}(r) &= s^{\text{bt}}(r) + s^{\text{bb}}(r) = \sigma_{rz}^{\text{core}}(r) + s^{\text{bb}}(r) \end{aligned}$$

Plugging these into equations (131) and (132) leads to, for the top facesheet,

$$\begin{aligned} \int_{\Omega_0} \left[N_{rr}^{\text{top}} \frac{d\delta u_{0r}^{\text{top}}}{dr} + \left\{ \frac{N_{\theta\theta}^{\text{top}}}{r} + \sigma_{rz}^{\text{core}} - s^{\text{tt}} \right\} \delta u_{0r}^{\text{top}} \right] d\Omega_0 &= \oint_{\Gamma_0} N_r^{\text{top}} \delta u_{0r}^{\text{top}} d\Gamma_0 \quad (135) \\ \int_{\Omega_0} \left[M_{rr}^{\text{top}} \frac{d^2\delta w_0^{\text{top}}}{dr^2} + \left\{ \frac{M_{\theta\theta}^{\text{top}}}{r} - f^{\text{top}} (s^{\text{tt}} - \sigma_{rz}^{\text{core}}) \right\} \frac{d\delta w_0^{\text{top}}}{dr} \right. \\ &\quad \left. - \left\{ \frac{C_{33}^{\text{core}}}{2c} (w_0^{\text{top}} - w_0^{\text{bot}}) - c \frac{d\sigma_{rz}^{\text{core}}}{dr} - c \frac{\sigma_{rz}^{\text{core}}}{r} \right\} \delta w_0^{\text{top}} \right] d\Omega_0 \\ &= \oint_{\Gamma_0} \left[M_r^{\text{top}} \frac{d\delta w_0^{\text{top}}}{dr} - Q_z^{\text{top}} \delta w_0^{\text{top}} \right] d\Gamma_0 \quad (136) \end{aligned}$$

and for the bottom facesheet

$$\begin{aligned} \int_{\Omega_0} \left[N_{rr}^{\text{bot}} \frac{d\delta u_{0r}^{\text{bot}}}{dr} + \left\{ \frac{N_{\theta\theta}^{\text{bot}}}{r} - \sigma_{rz}^{\text{core}} - s^{\text{bb}} \right\} \delta u_{0r}^{\text{bot}} \right] d\Omega_0 &= \oint_{\Gamma_0} N_r^{\text{bot}} \delta u_{0r}^{\text{bot}} d\Gamma_0 \quad (137) \\ \int_{\Omega_0} \left[M_{rr}^{\text{bot}} \frac{d^2\delta w_0^{\text{bot}}}{dr^2} + \left\{ \frac{M_{\theta\theta}^{\text{bot}}}{r} + f^{\text{bot}} (s^{\text{bb}} + \sigma_{rz}^{\text{core}}) \right\} \frac{d\delta w_0^{\text{bot}}}{dr} \right. \\ &\quad \left. + \left\{ \frac{C_{33}^{\text{core}}}{2c} (w_0^{\text{top}} - w_0^{\text{bot}}) + c \frac{d\sigma_{rz}^{\text{core}}}{dr} + c \frac{\sigma_{rz}^{\text{core}}}{r} \right\} \delta w_0^{\text{bot}} \right] d\Omega_0 \\ &= \oint_{\Gamma_0} \left[M_r^{\text{bot}} \frac{d\delta w_0^{\text{bot}}}{dr} - Q_z^{\text{bot}} \delta w_0^{\text{bot}} \right] d\Gamma_0 \quad (138) \end{aligned}$$

The governing ordinary differential equation for the core is

$$\begin{aligned} \frac{d^2\sigma_{rz}^{\text{core}}}{dr^2} + \frac{1}{r} \frac{d\sigma_{rz}^{\text{core}}}{dr} - \left(\frac{1}{r^2} + \frac{6C_{33}^{\text{core}} S_{55}^{\text{core}}}{c^2} \right) \sigma_{rz}^{\text{core}} \\ = -\frac{3C_{33}^{\text{core}}}{2c^3} \left[u_{0r}^{\text{top}} - u_{0r}^{\text{bot}} + (f^{\text{top}} + c) \frac{dw_0^{\text{top}}}{dr} + (f^{\text{bot}} + c) \frac{dw_0^{\text{bot}}}{dr} \right] \quad (139) \end{aligned}$$

Multiplying the equation with a test function and integration over the area Ω_0 yields, after an integration by parts, the equation:

$$\begin{aligned} \int_{\Omega_0} \left[\frac{d\sigma_{rz}^{\text{core}}}{dr} \frac{d\delta\sigma_{rz}}{dr} + \frac{1}{r} \left(\sigma_{rz}^{\text{core}} \frac{d\delta\sigma_{rz}}{dr} + \frac{d\sigma_{rz}^{\text{core}}}{dr} \delta\sigma_{rz} \right) + \left\{ \left(\frac{1}{r^2} + \frac{6C_{33}^{\text{core}} S_{55}^{\text{core}}}{c^2} \right) \sigma_{rz}^{\text{core}} \right. \right. \\ \left. \left. - \frac{3C_{33}^{\text{core}}}{2c^3} \left[u_{0r}^{\text{top}} - u_{0r}^{\text{bot}} + (f^{\text{top}} + c) \frac{dw_0^{\text{top}}}{dr} + (f^{\text{bot}} + c) \frac{dw_0^{\text{bot}}}{dr} \right] \right\} \delta\sigma_{rz} \right] d\Omega_0 \\ = \oint_{\Gamma_0} \left(\frac{d\sigma_{rz}^{\text{core}}}{dr} + \frac{\sigma_{rz}^{\text{core}}}{r} \right) \delta\sigma_{rz} d\Gamma_0 \end{aligned} \quad (140)$$

Equations (135), (136), (137), (138), and (140) form the system that has been discretized using the finite element approach.

We assume that the fields $u_{0r}^{\text{top}}, u_{0r}^{\text{bot}}, w_0^{\text{top}}, w_0^{\text{bot}}, \sigma_{rz}^{\text{core}}$ can be expressed as

$$\begin{aligned} u_{0r}^{\text{top}}(r) &= \sum_{i=1}^{nu} u_i^{\text{top}} N_i^u(r); \quad u_{0r}^{\text{bot}}(r) = \sum_{i=1}^{nu} u_i^{\text{bot}} N_i^u(r) \\ w_0^{\text{top}}(r) &= \sum_{i=1}^{nw} w_i^{\text{top}} N_i^w(r); \quad w_0^{\text{bot}}(r) = \sum_{i=1}^{nw} w_i^{\text{bot}} N_i^w(r) \\ \sigma_{rz}^{\text{core}}(r) &= \sum_{i=1}^{ns} \sigma_i N_i^s(r) \end{aligned} \quad (141)$$

where nu, nw, ns are the number of nodes and $N_i^{u,w,s}$ are the basis functions that are required to represent the field variables. Then, the stress and stress couple resultants can be expressed as

$$\begin{aligned} N_{rr}^{\text{top}} &= \sum_{j=1}^{nu} \left[A_{11}^{\text{top}} \frac{dN_j^u}{dr} + A_{12}^{\text{top}} \frac{N_j^u}{r} \right] u_j^{\text{top}}; \quad N_{\theta\theta}^{\text{top}} = \sum_{j=1}^{nu} \left[A_{12}^{\text{top}} \frac{dN_j^u}{dr} + A_{11}^{\text{top}} \frac{N_j^u}{r} \right] u_j^{\text{top}} \\ N_{rr}^{\text{bot}} &= \sum_{j=1}^{nu} \left[A_{11}^{\text{bot}} \frac{dN_j^u}{dr} + A_{12}^{\text{bot}} \frac{N_j^u}{r} \right] u_j^{\text{bot}}; \quad N_{\theta\theta}^{\text{bot}} = \sum_{j=1}^{nu} \left[A_{12}^{\text{bot}} \frac{dN_j^u}{dr} + A_{11}^{\text{bot}} \frac{N_j^u}{r} \right] u_j^{\text{bot}} \\ M_{rr}^{\text{top}} &= - \sum_{j=1}^{nw} \left[D_{11}^{\text{top}} \frac{d^2 N_j^w}{dr^2} + \frac{D_{12}^{\text{top}}}{r} \frac{dN_j^w}{dr} \right] w_j^{\text{top}}; \quad M_{\theta\theta}^{\text{top}} = - \sum_{j=1}^{nw} \left[D_{12}^{\text{top}} \frac{d^2 N_j^w}{dr^2} + \frac{D_{11}^{\text{top}}}{r} \frac{dN_j^w}{dr} \right] w_j^{\text{top}} \\ M_{rr}^{\text{bot}} &= - \sum_{j=1}^{nw} \left[D_{11}^{\text{bot}} \frac{d^2 N_j^w}{dr^2} + \frac{D_{12}^{\text{bot}}}{r} \frac{dN_j^w}{dr} \right] w_j^{\text{bot}}; \quad M_{\theta\theta}^{\text{bot}} = - \sum_{j=1}^{nw} \left[D_{12}^{\text{bot}} \frac{d^2 N_j^w}{dr^2} + \frac{D_{11}^{\text{bot}}}{r} \frac{dN_j^w}{dr} \right] w_j^{\text{bot}} \end{aligned} \quad (142)$$

and the momentum balance equations can be written as

$$\int_{\Omega_0} \left[N_{rr}^{\text{top}} \frac{dN_i^u}{dr} + \left\{ \frac{N_{\theta\theta}^{\text{top}}}{r} + \sum_{k=1}^{ns} \sigma_k N_k^s \right\} N_i^u \right] d\Omega_0 = \int_{\Omega_0} s^{\text{tt}} N_i^u d\Omega_0 + \oint_{\Gamma_0} N_r^{\text{top}} N_i^u d\Gamma_0 \quad (143)$$

$$\int_{\Omega_0} \left[N_{rr}^{\text{bot}} \frac{dN_i^u}{dr} + \left\{ \frac{N_{\theta\theta}^{\text{bot}}}{r} - \sum_{k=1}^{ns} \sigma_k N_k^s \right\} N_i^u \right] d\Omega_0 = \int_{\Omega_0} s^{\text{bb}} N_i^u d\Omega_0 + \oint_{\Gamma_0} N_r^{\text{bot}} N_i^u d\Gamma_0 \quad (144)$$

$$\begin{aligned} \int_{\Omega_0} \left[M_{rr}^{\text{top}} \frac{d^2 N_i^w}{dr^2} + \left\{ \frac{M_{\theta\theta}^{\text{top}}}{r} + f^{\text{top}} \sum_{k=1}^{ns} \sigma_k N_k^s \right\} \frac{dN_i^w}{dr} - \left\{ \frac{C_{33}^{\text{core}}}{2c} \sum_{k=1}^{nu} (w_k^{\text{top}} - w_k^{\text{bot}}) N_k^w \right. \right. \\ \left. \left. - c \sum_{k=1}^{ns} \left(\frac{dN_k^s}{dr} + \frac{N_k^s}{r} \right) \sigma_k \right\} N_i^w \right] d\Omega_0 \\ = \int_{\Omega_0} f^{\text{top}} s^{\text{tt}} \frac{dN_i^w}{dr} d\Omega_0 + \oint_{\Gamma_0} \left[M_r^{\text{top}} \frac{dN_i^w}{dr} - Q_z^{\text{top}} N_i^w \right] d\Gamma_0 \end{aligned} \quad (145)$$

$$\begin{aligned} \int_{\Omega_0} \left[M_{rr}^{\text{bot}} \frac{d^2 N_i^w}{dr^2} + \left\{ \frac{M_{\theta\theta}^{\text{bot}}}{r} + f^{\text{bot}} \sum_{k=1}^{ns} \sigma_k N_k^s \right\} \frac{dN_i^w}{dr} + \left\{ \frac{C_{33}^{\text{core}}}{2c} \sum_{k=1}^{nu} (w_k^{\text{top}} - w_k^{\text{bot}}) N_k^w \right. \right. \\ \left. \left. + c \sum_{k=1}^{ns} \left(\frac{dN_k^s}{dr} + \frac{N_k^s}{r} \right) \sigma_k \right\} N_i^w \right] d\Omega_0 \\ = - \int_{\Omega_0} f^{\text{bot}} s^{\text{bb}} \frac{dN_i^w}{dr} d\Omega_0 + \oint_{\Gamma_0} \left[M_r^{\text{bot}} \frac{dN_i^w}{dr} - Q_z^{\text{bot}} N_i^w \right] d\Gamma_0 \end{aligned} \quad (146)$$

$$\begin{aligned} \int_{\Omega_0} \left[\sum_{j=1}^{ns} \left\{ \left(\frac{dN_j^s}{dr} + \frac{N_j^s}{r} \right) \frac{dN_i^s}{dr} + \frac{dN_j^s}{dr} \frac{N_i^s}{r} + \left(\frac{1}{r^2} + \frac{6C_{33}^{\text{core}} S_{55}^{\text{core}}}{c^2} \right) N_j^s N_i^s \right\} \sigma_j \right. \\ \left. - \frac{3C_{33}^{\text{core}}}{2c^3} \left[\sum_{k=1}^{nu} N_k^u (u_k^{\text{top}} - u_k^{\text{bot}}) + \sum_{k=1}^{nw} \frac{dN_k^w}{dr} \left\{ (f^{\text{top}} + c) w_k^{\text{top}} + (f^{\text{bot}} + c) w_k^{\text{bot}} \right\} \right] N_i^s \right] d\Omega_0 \\ = \oint_{\Gamma_0} \left(\frac{d\sigma_{rz}^{\text{core}}}{dr} + \frac{\sigma_{rz}^{\text{core}}}{r} \right) N_i^s d\Gamma_0 \end{aligned} \quad (147)$$

After plugging in the expressions for the resultant stress and stress couples, we can express the above equations in matrix form as

$$\begin{bmatrix} [\mathbf{K}_{uu}^{tt}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{K}_{us}^t] \\ [\mathbf{0}] & [\mathbf{K}_{ww}^{tt}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{K}_{ws}^t] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{K}_{uu}^{bb}] & [\mathbf{0}] & [\mathbf{K}_{us}^b] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{K}_{ww}^{bb}] & [\mathbf{K}_{ws}^b] \\ [\mathbf{K}_{su}^t] & [\mathbf{K}_{sw}^t] & [\mathbf{K}_{su}^b] & [\mathbf{K}_{sw}^b] & [\mathbf{K}_{ss}] \end{bmatrix} \begin{bmatrix} \mathbf{u}^{\text{top}} \\ \mathbf{w}^{\text{top}} \\ \mathbf{u}^{\text{bot}} \\ \mathbf{w}^{\text{bot}} \\ \boldsymbol{\sigma} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_u^{\text{top}} \\ \mathbf{f}_w^{\text{top}} \\ \mathbf{f}_u^{\text{bot}} \\ \mathbf{f}_w^{\text{bot}} \\ \mathbf{f}_\sigma \end{bmatrix} \quad (148)$$

4.1 Finite element basis functions

Note that the stiffness matrix is not symmetric. This system of equations is solved using COMSOLTM using quadratic shape functions for the u -displacement and the σ -stress and cubic

Hermite functions for the w -displacement, i.e., in each element $nu = ns = 3$, $nw = 4$, and

$$\begin{aligned} N_i^u(r) &= N_i^s(r) = \prod_{j=1, i \neq j}^3 \frac{r - r_j}{r_i - r_j} \\ N_1^w(r) &= 1 - 3r^2 + 2r^3; \quad N_2^w(r) = (r_2 - r_1)(r - 2r^2 + r^3) \\ N_3^w(r) &= 3r^2 - 2r^3; \quad N_4^w(r) = (r_2 - r_1)(-r^2 + r^3) \end{aligned} \quad (149)$$

4.2 Boundary conditions

The natural boundary conditions are

$$\begin{aligned} \text{Top facesheet : } & N_r^{\text{top}}, Q_z^{\text{top}}, M_r^{\text{top}} \\ \text{Bottom facesheet : } & N_r^{\text{bot}}, Q_z^{\text{bot}}, M_r^{\text{bot}} \\ \text{Core : } & \frac{d\sigma_{rz}^{\text{core}}}{dr} + \frac{\sigma_{rz}^{\text{core}}}{r} \end{aligned} \quad (150)$$

The essential boundary conditions are

$$\begin{aligned} \text{Top facesheet : } & u_{0r}^{\text{top}}, w_0^{\text{top}}, \frac{dw_0^{\text{top}}}{dr} \\ \text{Bottom facesheet : } & u_{0r}^{\text{bot}}, w_0^{\text{bot}}, \frac{dw_0^{\text{bot}}}{dr} \\ \text{Core : } & \sigma_{rz}^{\text{core}} \end{aligned} \quad (151)$$

Note that fixing the u_z displacement at the boundary of the core is equivalent to setting the natural boundary condition in the core to zero when $w_0^{\text{top}} = w_0^{\text{bot}} = 0$ at the boundary.

4.2.1 Through-the-thickness insert

The boundary conditions used for a simply-supported sandwich panel with a through-the-thickness insert are shown in Figure 3. The radius of the insert is r_i , that of the potting is r_p , and that of the panel is r_a . Therefore, for part of the panel, the potting is assumed to have the same behavior as the core.

For this situation, the boundary conditions at the left edge, $r = r_i$, are

$$\begin{aligned} u_{0r}^{\text{top}} &= u_{0r}^{\text{bot}} = 0; \quad M_r^{\text{top}} = M_r^{\text{bot}} = 0 \\ Q_z^{\text{top}} &= \frac{Q f^{\text{top}}}{f^{\text{top}} + 2c + f^{\text{bot}}}; \quad Q_z^{\text{bot}} = \frac{Q f^{\text{bot}}}{f^{\text{top}} + 2c + f^{\text{bot}}} \\ \sigma_{rz}^{\text{core}} &= \frac{Q}{A}, \end{aligned} \quad (152)$$

where $A = 2\pi r_i (f^{\text{top}} + 2c + f^{\text{bot}})$. These are applied to the two dimensional model as a constant pressure in the z direction, with $P = Q/A$.

At the right edge, $r = r_a$, the structure is simply supported, with the conditions:

$$w^{\text{top}} = w^{\text{bot}} = 0; \quad M_r^{\text{top}} = M_r^{\text{bot}} = 0; \quad N_r^{\text{top}} = N_r^{\text{bot}} = 0; \quad \frac{d\sigma_{rz}}{dr} + \frac{\sigma_{rz}}{r_a} = 0. \quad (153)$$

The support condition is applied to the two dimensional model by setting $w_0 = 0$ along the right

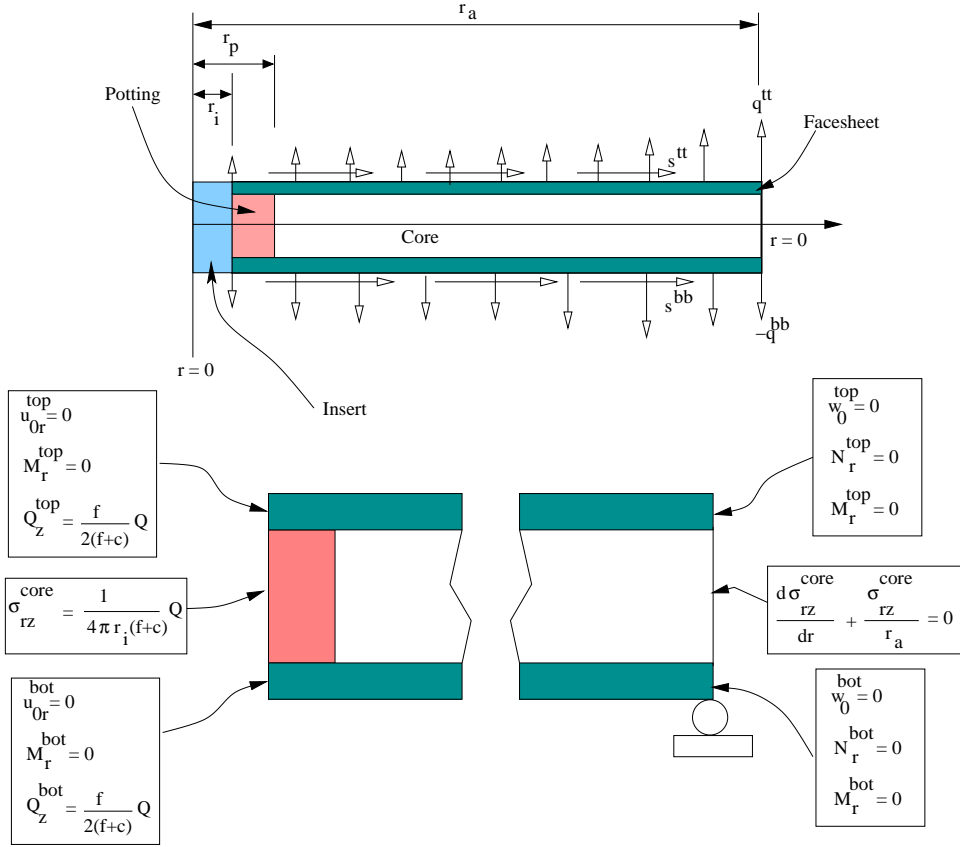


Figure 3 – Boundary conditions for a through-the-thickness insert in a simply supported sandwich panel.

edge.

4.2.2 Potted insert

The boundary conditions used for a simply-supported sandwich panel with a potted insert are shown in Figure 4. The radius of the insert is r_i , that of the potting is r_p , and that of the panel is r_a . The length of the insert is $2f_i$ and the thickness of the potting below the insert is $2c^-$. Therefore, the insert is being treated as a thin plate in the region above the potting and the potting is being treated as a material with features similar to the core.

To allow for the jump discontinuities on the two sides of the insert-facesheet interface, we define the quantities u_{0r}^- and u_{0r}^+ to be the u_{0r} displacements of the insert and the top facesheet, respectively. The locations where these quantities are evaluated are shown in Figure 4. Then the continuity of displacements requires that

$$u_{0r}^+ = u_{0r}^- - (f_i - f) \frac{dw_0^{\text{interface}}}{dr} \quad (154)$$

There is also a jump in the shear stress in the two sections of the potting to the left and right of the interface. Let these quantities be σ_{rz}^- and σ_{rz}^+ . We assume that the average force at the interface is balanced, i.e.,

$$c^- \sigma_{rz}^- = c \sigma_{rz}^+ . \quad (155)$$

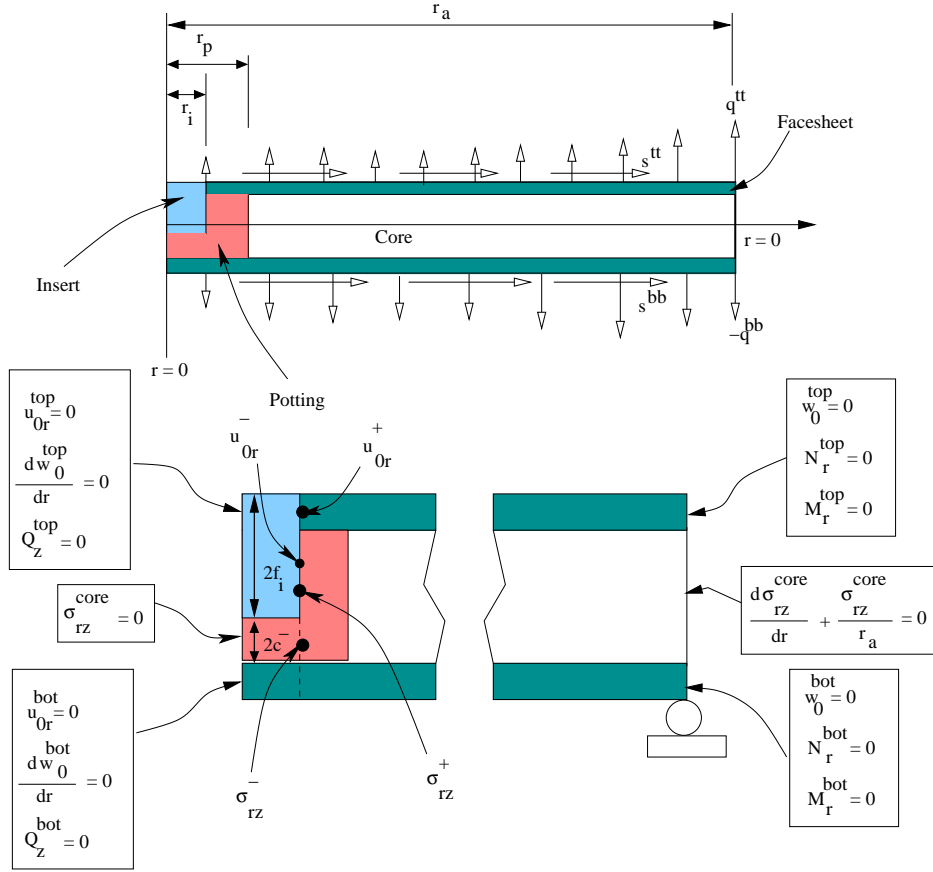


Figure 4 – Boundary conditions for a potted insert in a simply supported sandwich panel.

The boundary conditions at $r = 0$ are

$$u_{0r}^{top} = u_{0r}^{bot} = 0; \quad \frac{dw_0^{top}}{dr} = \frac{dw_0^{bot}}{dr} = 0; \quad Q_z^{top} = Q_z^{bot} = 0; \quad \sigma_{rz}^{core} = 0 \quad (156)$$

The simply-supported boundary at $r = r_a$ once again requires that

$$w_0^{top} = w_0^{bot} = 0; \quad M_r^{top} = M_r^{bot} = 0; \quad N_r^{top} = N_r^{bot} = 0; \quad \frac{d\sigma_{rz}^{core}}{dr} + \frac{\sigma_{rz}^{core}}{r_a} = 0. \quad (157)$$

5 Model Test Cases: FRP Sandwich

In order to validate the one dimensional approximation, the results for test cases are compared with the results generated by a two dimensional axisymmetric model. In each test case, a rigid, through the thickness insert applies a vertical compression load of $Q = 1000\text{N}$ to a simply supported sandwich panel.

The facesheets are assumed to be isotropic, i.e.,

$$\begin{aligned} A_{11} &= 2 f C_{11} = 2f \frac{E}{1-\nu^2}; & A_{12} &= 2 f C_{12} = 2f \frac{\nu E}{1-\nu^2} \\ D_{11} &= \frac{2 f^3}{3} C_{11} = \frac{2f^3}{3} \frac{E}{1-\nu^2}; & D_{12} &= \frac{2 f^3}{3} C_{12} = \frac{2f^3}{3} \frac{\nu E}{1-\nu^2} \end{aligned} \quad (158)$$

where E is the Young's modulus and ν is the Poisson's ratio.

The core moduli are given by

$$C_{33} = \frac{E_h}{1-\nu_h^2}; \quad C_{55} = 2 G_h \quad (159)$$

where E_h, G_h, ν_h are the Young's modulus, shear modulus, and the Poisson's ratio of the core. The potting moduli are also computed in a manner similar to those of the core.

5.1 Example 1: Stiff facesheets

The first example problem is taken from [7], with the parameters given in Table 1. Figure 5(a) compares the resulting out of plane displacements from the sandwich theory and the two dimensional axisymmetric simulations. While there is a small amount of disagreement in the potting region, the overall results match up well. The radial displacement, u_r , is shown in figure 5(b), and these results match as well. Core shear stresses and transverse stresses at the bottom of the core are shown in Figure 6. The stresses match reasonably well too.

Table 1 – Geometric and material parameters for Example 1

Geometry (mm):	$r_i = 7.0$	$r_p = 10.0$	$r_a = 60.0$	$c = 10.0$	$f^{\text{top}} = 0.1$	$f^{\text{bot}} = 0.1$
Face Sheets (GPa) :	$E_1 = 71.5$	$G_1 = 27.5$				
Potting (GPa) :	$E_p = 2.5$	$G_p = 0.93$				
Honeycomb (MPa) :	$E_h = 310$	$G_h = 138$				

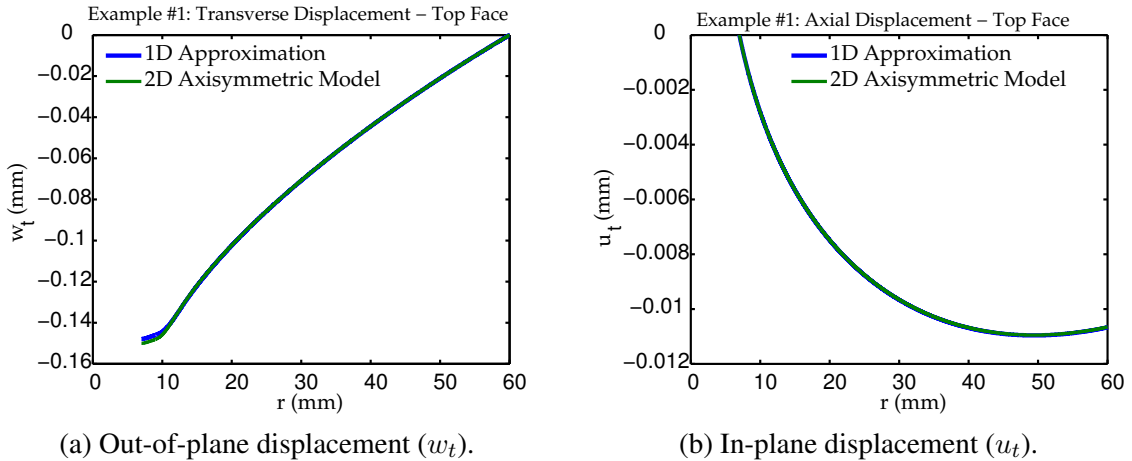


Figure 5 – Comparisons of displacements from one-dimensional and two-dimensional finite element simulations for the model in Table 1.

5.2 Example 2: Soft facesheets

The second example problem is taken from [8], with the parameters given in Table 2. Once again, figure 7(a) shows the out of plane displacements given by the sandwich theory and the axisymmetric

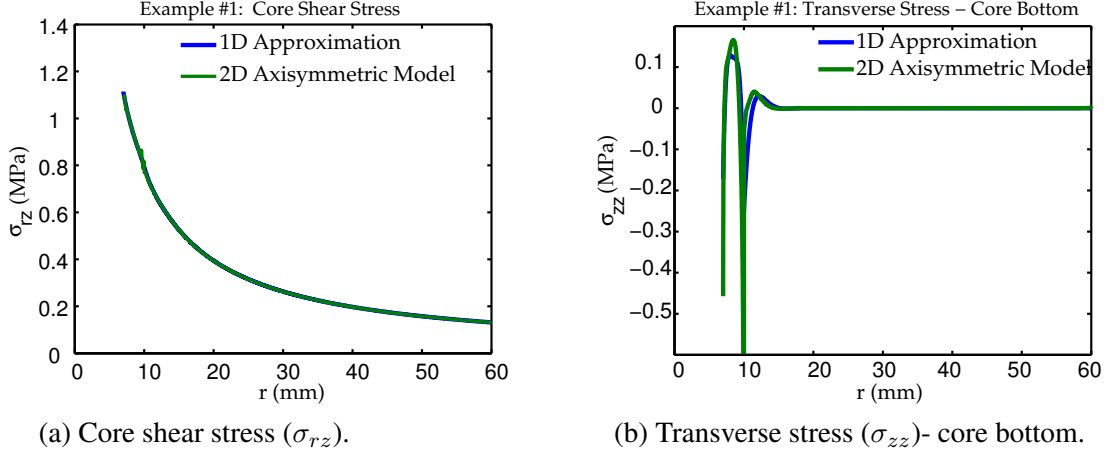


Figure 6 – Comparisons of stresses from one-dimensional and two-dimensional finite element simulations for the model in Table 1.

simulations, and figure 7(b) gives the radial displacements. As in the first example, the results match reasonably well, suggesting that the sandwich theory captures the important physics of the problem. The stresses shown in Figure 8 also show that the one- and two-dimensional models predict similar results. The values of transverse stress and displacement at the bottom of the core are shown in Figure 9.

Table 2 – Geometric and material parameters for Example 2

Geometry (mm):	$r_i = 10.0$	$r_p = 30.0$	$r_a = 150.0$	$c = 5.0$	$f^{\text{top}} = 0.5$	$f^{\text{bot}} = 0.1$
Face Sheets (GPa):	$E_1 = 40.0$	$G_1 = 14.8$				
Potting (GPa):	$E_p = 2.5$	$G_p = 0.93$				
Honeycomb (MPa):	$E_h = 310$	$G_h = 138$				

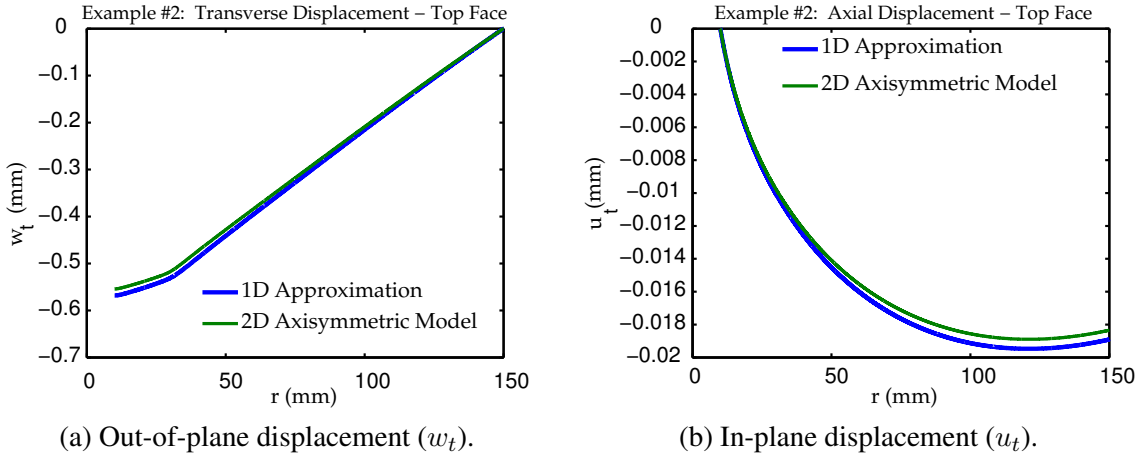


Figure 7 – Comparison of displacements from one-dimensional and two-dimensional finite element simulations for the model in Table 2.

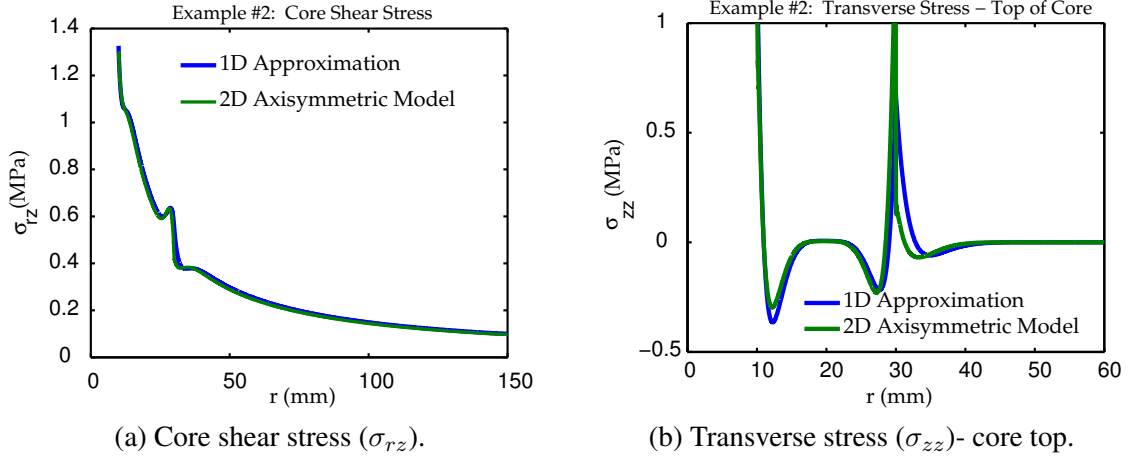


Figure 8 – Comparison of stresses from one-dimensional and two-dimensional finite element simulations for the model in Table 2.

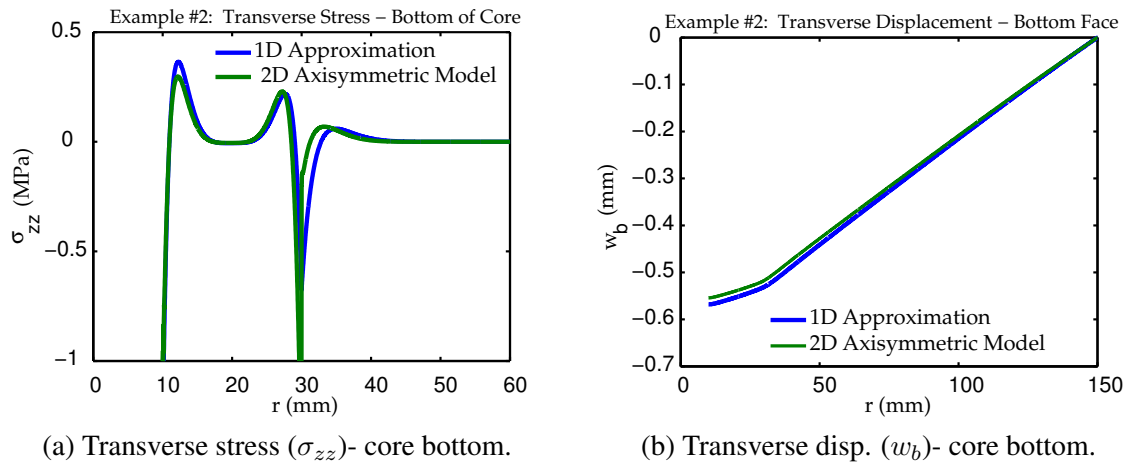


Figure 9 – Comparison of stresses and displacements at the bottom of the core from one-dimensional and two-dimensional finite element simulations for the model in Table 2.

6 Summary and Conclusions

A detailed one-dimensional theory for sandwich panels with inserts has been derived. The approach follows that used by Thomsen [8]. The models have been discretized using a finite element approach. The one-dimensional model produces results that are close to those of a two-dimensional axisymmetric finite element model. Both models assume that the core is homogeneous, indicating that the one-dimensional model might be well suited for small deformations of sandwich specimens with foam cores. Further work is needed to find nonlinear one-dimensional models of sandwich panels with inserts.

Acknowledgments

This work was funded by a subcontract of the University of Auckland Research for Industry grant “Advanced Composite Structures” from the New Zealand Foundation for Research, Science, and Technology.

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